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Projekt 7.8 Finding mean and average values using integration

(Denne tekst er hentet fra et engelsk on-line universitetskursus om integration. En del steder henvises til andre "units", hvor man har lært forskellige ting. Eksempler og øvelser kan læses og løses uden at gå til disse. Henvises der til en "Handbook" for at finde integraler, så anvender du dit værktøjsprogram. Kun i ganske få lande er det tilladt at anvende sådanne, i stedet lærer de at slå op i store tabelværker.)

In a discrete model an average is found by adding up a series of numbers and dividing the total by the number of numbers. This method can be carried over to continuous models, but, in this case, the average involves an integral rather than a sum. Models which give the distribution of ages can be used to calculate mean or average ages. The process of finding the mean age from discrete models can also be carried over in the limiting case to continuous models.

1. The average value of a continuous function

In Unit 5 there was a discussion of the difference between instantaneous velocity and average velocity. If you know the instantaneous velocity between two particular times, it is possible to use this to find the average velocity between these times.

If the instantaneous velocity, *v*, at time *t* is a function of *t* between times t_1 and t_2, then the average velocity between these two times is

$$\frac{\text{total distance travelled between } t=t_1 \text{ and } t=t_2}{\text{total time}}$$

Unit 9 has shown you that if you know *v* as a function of time, or if you have adopted a satisfactory model of *v* as a function of *t,* you can find the total distance travelled by evaluating

$$\int_{t_1}^{t_2} v\,dt$$

and the total time is of course just $t_2 - t_1$. So if I denote the average velocity by $\bar{v}$ then

$$\bar{v} = \frac{\int_{t_1}^{t_2} v\,dt}{t_2 - t_1}$$

Is this equation dimensionally balanced?

The left-hand side has dimensions of [length] ÷ [time]. The right-hand side has dimensions of

$$\frac{[length]}{[time]} \times [time] \times \frac{1}{[time]}$$

which is [length] ÷ [time]. So the equation is dimensionally balanced.

Example 1

An object moves and its average velocity at time *t* is modelled by

$$v = a \cdot t^2 \quad (0 \le t \le T)$$

What is its average velocity over this range of time?

$$\bar{v} = \frac{\int_0^T a \cdot t^2\,dt}{T-0} = \frac{1}{T} \cdot \left[\frac{1}{3} \cdot a \cdot t^3\right]_0^T = \frac{1}{3} \cdot a \cdot T^2$$

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Øvelse 1

An object moves and its instantaneous velocity at time t is modelled by

$$v = v_0 + a \cdot t^{1/2}$$

What is its average velocity between $t = 0$ and $t = T$?
The same method can be used for variables other than velocity.

One example is the calculation of the average power supplied to a domestic electric appliance. The power supplied by the normal mains in the United Kingdom is designed such that it varies according to the formula

$$p = p_0 \cdot \sin^2(\omega \cdot t)$$

Figure 9 shows a sketch of p as t varies. The value of ω is about 300 radians per second.

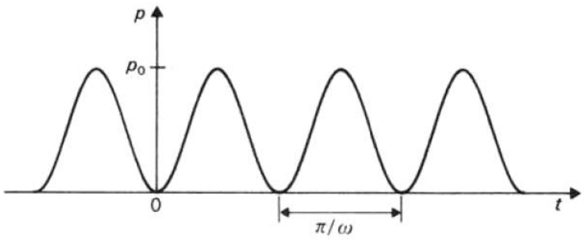

Figure 9

Domestic appliances are given a power rating which is the *average* power they consume (for instance a 60 watt lamp consumes an average power of 60 watts: a 3 kilowatt heater consumes an average power of 3 kilowatts), though of course they must be built to withstand repeated short periods at the peak power.

The average power, $\bar{p}$, can be found by calculating the average over just one cycle, since the symmetry of the graph means that the average over many cycles is the same as the average over one cycle. One cycle is of duration π/ω so

$$\bar{p} = \frac{\int_0^{\pi/\omega} p\,dt}{\pi/\omega - 0}$$
$$= \frac{\omega}{\pi} \cdot \int_0^{\pi/\omega} p_0 \cdot \sin^2(\omega \cdot t)\,dt$$

The table of integrals in your *Handbook* does not give a standard integral for $\sin^2(\omega \cdot t)$ or for any function that looks similar to it. The only trigonometric functions given are $\sin(a \cdot t)$ and $\cos(a \cdot t)$.

Find a formula in Unit 3 or in your *Handbook* that relates $\sin^2(\alpha)$ to $\cos(2\alpha)$.

$$\cos(2\alpha) = 1 - 2\sin^2(\alpha)$$

Rearranging this formula gives

$$\sin^2(\alpha) = \frac{1}{2} \cdot (1 - \cos(2\alpha))$$

We can therefore replace $\sin^2(\omega \cdot t)$ in the integral for $\bar{p}$ by $\frac{1}{2} \cdot (1 - \cos(2\omega \cdot t))$, so

$$\bar{p} = \frac{\omega \cdot p_0}{\pi} \cdot \int_0^{\pi/\omega} \frac{1}{2} \cdot (1 - \cos(2\omega \cdot t))\,dt$$
$$= \frac{\omega \cdot p_0}{2\pi} \cdot \int_0^{\pi/\omega} (1 - \cos(2\omega \cdot t))\,dt$$

I can integrate both 1 and $\cos(\omega \cdot t)$ to give

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$$\bar{p} = \frac{\omega \cdot p_0}{2\pi} \cdot \left[t - \frac{1}{2\omega} \cdot \sin(2\omega \cdot t) \right]_0^{\pi/\omega}$$

$$= \frac{\omega \cdot p_0}{2\pi} \cdot \left[\left(\frac{\pi}{\omega} - \frac{1}{2\omega} \cdot \sin(2\pi) \right) - \left(0 - \frac{1}{2\omega} \cdot \sin(0) \right) \right]$$

But $\sin(2\pi) = \sin(0) = 0$, so

$$\bar{p} = \frac{\omega \cdot p_0}{2\pi} \times \frac{\pi}{\omega} = \frac{p_0}{2}$$

So the average power is half of the peak power. Thus an electric light bulb with an average power consumption of 100 watts actually has a *peak* power consumption of 200 watts. This peak consumption occurs 100 times a second. This is too fast for the normal eye to be able to detect the variation in brightness, though the variation does occur.

A similar manipulation to the one I used to find the integral of $\sin^2(\omega \cdot t)$ can be used to integrate $\cos^2(\omega \cdot t)$.
How would you rewrite $\cos^2(\omega \cdot t)$ in order to evaluate $\int \cos^2(\omega \cdot t)\, dt$?

$$\cos^2(\omega \cdot t) = \frac{1}{2} \cdot (1 + \cos(2\omega \cdot t))$$

The method for finding the average velocity over a period of time can be generalized for finding the average value of any function $y = f(x)$ over an interval of values of x. This average $\bar{y}$, between x_1 and x_2 is given by

$$\bar{y} = \frac{\int_{x_1}^{x_2} y\, dx}{x_2 - x_1}$$

Example 2

The height h of the cross-section of the water surface between two glass plates shown in Figure 10 can be modelled by the equation

$$h = h_0 + \alpha \cdot x^2 \quad (-a \le x \le a)$$

where x is the distance from a line midway between the plates. What is the average height of the surface predicted by the model?

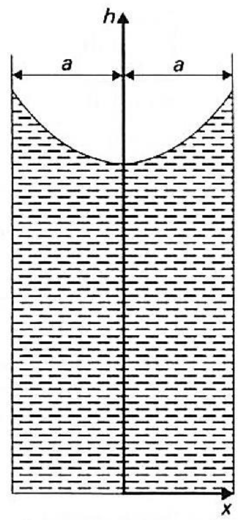

Figure 10

$$\bar{h} = \frac{\int_{-a}^{a} h_0 + \alpha \cdot x^2\, dx}{a - (-a)}$$

$$= \frac{1}{2a} \cdot \left[h_0 \cdot x + \frac{1}{3} \cdot \alpha \cdot x^3 \right]_{-a}^{a}$$

$$= \frac{1}{2a} \cdot \left[h_0 \cdot a + \frac{1}{3} \cdot \alpha \cdot a^3 - h_0 \cdot (-a) - \frac{1}{3} \cdot \alpha \cdot (-a)^3 \right]$$

$$= \frac{1}{2a} \cdot \left[2h_0 \cdot a + \frac{2}{3} \cdot \alpha \cdot a^3 \right]$$

$$= h_0 + \frac{1}{3} \cdot \alpha \cdot a^3$$

So the average height predicted is

$$h_0 + \frac{1}{3} \cdot \alpha \cdot a^3$$

Øvelse 2

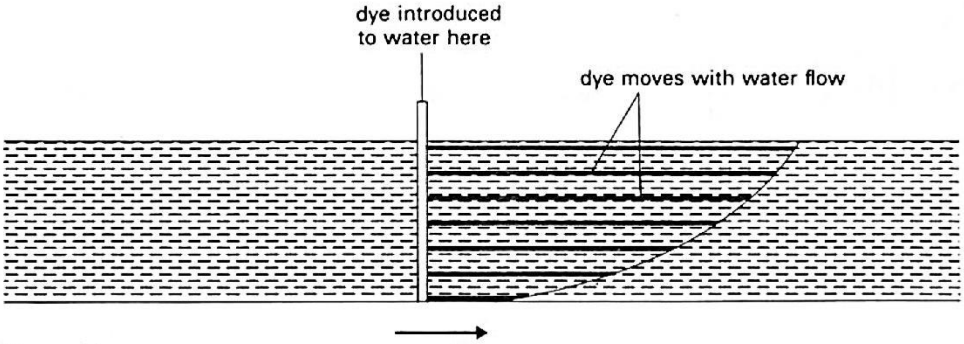

Figure 11

In TV5 you were told that the average velocity of the water flowing in a vertical slice of a river is about the same as the velocity 0.6 of the way down (or 0.4 of the way up). This result is based on experimental evidence produced in tanks like that shown in Figure 11. Spots of dye are introduced into the water at different depths at one instant of time and their positions recorded at subsequent times. Data from this type of experiment shows that the velocity *v* of the water at a height *h* units from the bottom of the tank can be reasonably well modelled by the function

$$v = v_{max} \cdot \left(1 - a \cdot \exp(-k \cdot h)\right) \quad \left(0 \le h \le h_{max}\right)$$ [1]

(a) Find an expression for $\bar{v}$ in terms of v_{max}, using the following values of the parameters

$$a = 1, \quad k = 0.5 \quad \text{and} \quad h_{max} = 7$$

(b) Find the value of *h* for which $v = \bar{v}$.

(c) What fraction of the way up corresponds to this value of *h?*
What fraction of the way down is $v = \bar{v}$?

Now use these ideas about averages in a town planning model in order to predict an ‘average' type of trip to a shopping centre.

Øvelse 3

A roughly circular town has a central business zone in which very few people live and a residential zone with a population density which can be modelled by

$$D = k_1 / r^2 \text{ people per square kilometre} \quad \left(\frac{1}{2} \le r \le 5\right)$$

at a distance *r* km from the town centre.

Each person can be assumed to make k_2 / r^2 trips per week to and from a shopping centre located at $r = 0$. In this model k_1 and k_2 are constant parameters.

(a) How many people live in an annulus of radius *r* km and width Δr km, where Δr is small?
(b) How many shopping trips are made per week by the people in this annulus?
(c) What is the total distance travelled by these people per week in doing their shopping?
(d) How many trips are made per week by all the people in the residential zone from $r = \frac{1}{2}$ to $r = 5$?
(e) What is the total distance travelled per week by all the people in this residential zone in doing their shopping?

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[1] In an experiment of this kind performed by George Parr of Nottingham University. the data fits the above equation well

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(f) What is the average distance they travel each week according to the model?

Sometimes a variable may be modelled in more than one way. It can be regarded as being a function of different variables. For instance, as a car collides with an obstacle and is crushed, the force on it may be modelled as a function of distance – in which case an average force with respect to distance can be found.

Alternatively it can be modelled as a function of time — in which case an average force with respect to time can be found. These two averages are not, in general, equal and so it is necessary to make clear which average is being used in any given circumstances.

2. The mean value of a distribution

So far I have looked at how to find the average value of a function: the average of a velocity where $v = f(t)$; or the average value of power where $p = g(t)$. However, there is a different idea of average.

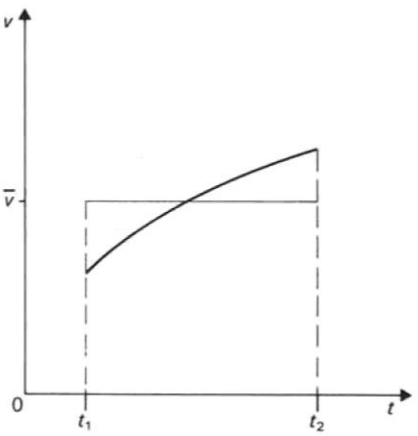

Figure 12 (a) Average velocity

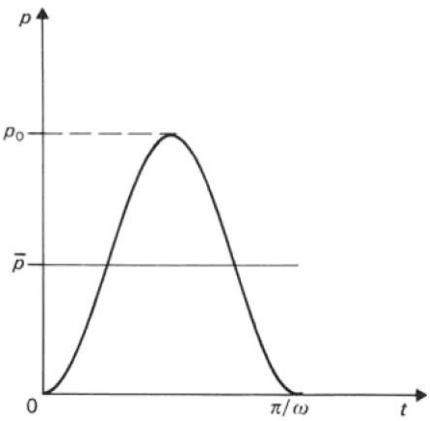

Figure 12 (b) Average power

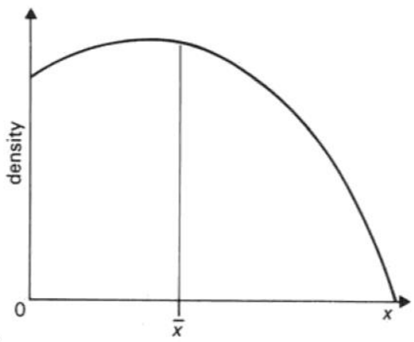

Figure 12 (c) Mean value

If you look at Figure 12 you will see the difference between finding an average velocity or average power (Figures 12(a) and 12(b)) and finding the average age for an age distribution (Figure 12(c)). This average is called the *mean value* of a distribution, to distinguish it from other types of average. If I am trying to find the average or mean value for the age distribution of a group of people, the average I want is the average of the independent variable plotted *along the horizontal axis* (Figure 12(c)). In the other examples, the average was that of the dependent variable plotted *up the vertical axis* (Figures 12(a) and (b)). Hence I cannot expect to use the same formula to find the mean value of a distribution.

Øvelse 4

In each of the following cases decide whether the required average under consideration is of the dependent or independent variable, by sketching the function given. *Do not evaluate the average.*

(a) The average height where the distribution of heights h m of people in a particular area is modelled by the density function

$$D = 0.306 \cdot \left(1 - \sin\left(\frac{\pi \cdot h}{4}\right)\right) \quad (0 \leq h \leq 2)$$

(b) The average height of a ball whose instantaneous height h m after a time t seconds is modelled by

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$$h = 5 - 5t^2 \quad (0 \le h \le 5)$$

(c) The average height of a road above sea level where the height *h* m of a point *D* m from a particular town is modelled by

$$h = 300 - 200 \cdot \exp(-0.1D) \quad (0 \le D \le 10)$$

The mean value of a distribution can be illustrated more easily by first looking at a small discrete distribution. If I had a group of 10 children such that two of them were 2 years old, three of them were 3 years old, three were 4 years old and the other two were 5 years old, I could calculate their average age in several ways. One way is to add up all their ages and divide by 10.

$$\frac{2+2+3+3+3+4+4+4+5+5}{10}$$

Thus the mean age, $\bar{t}$, is given by

$$\bar{t} = \frac{\text{total of all the children's age}}{\text{total number of children}}$$

It might be easier though to write it as

$$\bar{t} = \frac{2\times 2 + 3\times 3 + 3\times 4 + 2\times 5}{10}$$

In this case I have added up the number who are 2 times 2, the number who are 3 times 3, the number who are 3 times 4 and the number who are 2 times 5. I can express this as

$$\bar{t} = \frac{\sum_{t=2}^{5} N_t \cdot t}{10}$$

where N_t is the number who are *t* years old.

However, if I divide through each item in the sum individually by 10, I get another useful result

$$\bar{t} = 0.2\times 2 + 0.3\times 3 + 0.3\times 4 + 0.2\times 5$$

or in the same notation as equation (2)

$$\bar{t} = \sum_{t=2}^{5} (N_t / 10) \cdot t$$

In the latter case I have added up (the *fraction* of children who are 2) x 2, (the *fraction* who are 3) x 3, (the *fraction* who are 4) x 4 and (the *fraction* who are 5) x 5. I can express this as

$$\bar{t} = \sum_{t=2}^{5} f_t \cdot t$$

where *f,* is the fraction who are t years old.

Now I want to extend this idea to a continuous distribution.

Figure 13 shows the continuous distribution of the ages, *t* years, of a group of people. I shall divide this distribution into strips each of width Δt and examine one strip, which begins at age *t* years and has a corresponding density value *D.*
The fraction of all the people represented by this strip is given by its area

$$D \cdot \Delta t$$

If there are *P* people in all, the *number* of people in this strip is

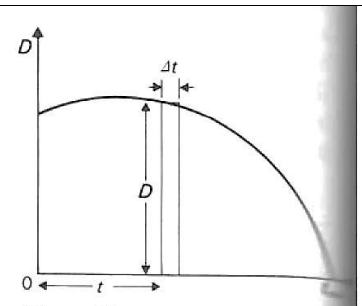

Figure 13

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$P \cdot D \cdot \Delta t$	

Since the strip is narrow, I can take it that all the people in this strip have age *t* years and so the total of all the ages of all the people in this strip is

$$P \cdot t \cdot D \cdot \Delta t$$

The total of all ages of all the people in the group is

$$\sum P \cdot t \cdot D \cdot \Delta t$$

summed over all the strips in the distribution, between $t = 0$ and $t = t_{max}$ (where t_{max} years is the maximum age).

The mean age $\bar{t}$ years is given by

$$\bar{t} = \frac{\text{total ages of all people}}{\text{total number of people}}$$
$$= \sum \frac{P \cdot t \cdot D \cdot \Delta t}{P}$$

summed over all strips between 0 and $t = t_{max}$.

Since the constant *P* occurs in the numerator and denominator it can be divided out, so

$$\bar{t} = \sum t \cdot D \cdot \Delta t$$

summed over all strips between $t = 0$ and t_{max}.

In the limit as the number of strips tends to infinity, and Δt tends to 0, the sum will become an integral between the limits 0 and t_{max}.

$$\bar{t} = \int_0^{t_{max}} t \cdot D \, dt$$

This formula allows me to calculate the mean age for a given population distribution, provided I know, or can model *D* as a function of *t.*

Eksempel 3

Find the average age of the population whose age distribution is given in SAQ 5 namely

$$D = \frac{3}{640000} \cdot \left(3200 + 40 \cdot t - t^2\right) \quad \left(0 \le t \le 80\right)$$

The average is given by

$$\int_0^{80} t \cdot D \, dt$$

which is

$$\int_0^{80} \frac{3}{640000} \cdot \left(3200 \cdot t + 40 \cdot t^2 - t^3\right) dt$$
$$= \frac{3}{640000} \cdot \left[1600 \cdot t^2 + \frac{40}{3} \cdot t^3 - \frac{1}{4} \cdot t^4\right]_0^{80}$$
$$= \frac{3}{640000} \cdot \left[1600 \cdot (80)^2 + \frac{40}{3} \cdot (80)^3 - \frac{1}{4} \cdot (80)^4 - 0\right]$$
$$= 32$$

So the average age is 32 years.

Øvelse 5

In SAQ 4 you met a population distribution modelled by

$$D = \frac{3}{800} \cdot \left(6 - \frac{t}{15}\right) \quad (0 \le t \le 80)$$

where *t* is the age in years.

(a) Why would you not expect the mean age to be 40?
(b) What is the mean age?

3. Summary

When you are calculating an average value it is essential to ask yourself: am I calculating the average value of a function or am I calculating the mean value of a distribution? A graph as in Figure 12 may help you. The formula you use will depend on which of these two things you are trying to do.

The average value of velocity, *v*, over the range $t = t_1$ to $t = t_2$ is given by

$$\bar{v} = \frac{\int_{t_1}^{t_2} v\,dt}{t_2 - t_1}$$

and in general the average value of a function *y* of a variable *x* between x_1 and x_2 will be given by

$$\bar{y} = \frac{\int_{x_1}^{x_2} y\,dx}{x_2 - x_1}$$

The mean value of a variable *t* over a range of values of *t* from t_1 to t_2 is given by

$$\bar{t} = \int_{t_1}^{t_2} t \cdot D\,dt$$

where *D* is the density function for the distribution of *t.*