
FRANCISCUS PRÆPOSITUS S. SALVATORIS POLLINGÆ Æ.1744 ✱
AD BIBLIOTHECAM IBIDEM.

Jones

Mathesis. Systemata & methodi: 110.

Synopsis Palmariorum Matheseos:

OR, A

New Introduction

TO THE

MATHEMATICS:

Containing the

PRINCIPLES

OF

Arithmetic & Geometry

DEMONSTRATED,

In a Short and Easie Method;

WITH

Their Application to the most Useful Parts thereof: As, Resolving of *Equations*, *Infinite Series*, Making the *Logarithms*; *Interest*, *Simple* and *Compound*; The Chief Properties of the *Conic Sections*; Mensuration of *Surfaces* and *Solids*; The Fundamental Precepts of *Perspective*; *Trigonometry*; The *Laws* of *Motion* apply'd to *Mechanic Powers*, *Gunnery*, &c.

Design'd for the Benefit, and adapted to the Capacities of BEGINNERS.

By *W.* *JONES*.

LONDON: Printed by *J. Matthews* for *Jeff. Wale* at the *Angel* in St. *Paul's* Church-Yard, 1706.

TO THE

HONOURABLE

William Lowndes, Esq;

SECRETARY

TO THE

TREASURY.

SIR,

IN this Attempt, I fear, I have incurr'd a double Censure: One by Inscribing a Small Tract

 of

of ARITHMETIC, to ſo Great a Maſter in Numbers; And another by throwing in my poor endeavors among the Public Affairs, and Crowd of Tranſactions of the Higheſt Concern, which Continually paſs thro' Your Hands.

I ſhou'd, therefore, offer at an Apology for my preſumption; And entreat Your favorable Interpretation of my performances.

I might inform You too, Sir, of the Forwardneſs of my Zeal to produce ſomething more worthy Your Acceptance, by a future Progreſs in theſe Studies.

My Devotion would lead me farther: But remembring how Valuable every Moment muſt be to a Perſon in Your Station, and to One that ſo Faithfully, ſo Exactly diſcharges the Important Buſineſs of it; I only crave the

Honor

Honor to Declare, how Ambitious I am of ſhewing my juſt Eſteem and Reſpect, and how Proud to ſubſcribe my ſelf,

SIR,

Your moſt obedient,

moſt humble Servant,

W. Jones.

THE PREFACE.

THE greatest part of the following Sheets was drawn up, and put into this Form, for the Use of some Friends, who had neither Leisure, Conveniency, nor, perhaps, Patience to search into so many different Authors, and turn over so many tedious Volumes, as is unavoidably requir'd to make but a tolerable Progress in the Mathematics: And the Benefit receiv'd by the Method we have taken, encourag'd us to Publish the same, for the Use of Learners, whether inclin'd to the Study of them, or occasionally oblig'd to be Employ'd in the Practice of some part of this Science; whose Principles, therefore, they are concern'd to be well acquainted with, but are unwilling to be incumber'd with more than is just necessary. The Particulars insisted upon in this Treatise are as follow.

1. *The Fundamental Rules of* Numeral Computation *or* Common Arithmetic *in whole Numbers are briefly, but fully deliver'd; with the necessary Examples in the several Cases; not omitting the most Simple Compendiums in the Operations, that might be of any Use to Practitioners.*

2. *The First Principles of* Literal Computation, *usually called* Algebra, *are illustrated with Variety of Instances, and the Reasons of the several Operations.*

3. *The usual Methods, as well of Raising Simple* Powers, *as of Extracting the Roots of such* Powers, *with Examples both in Numbers and Species.*

4. *The Nature and Properties of* Proportion; *where we have bin designedly large, adapting, as far as possible, every thing to the Capacities of young Beginners, that they might not be to seek for their Chief Materials, in prosecuting these Studies: For the whole Body of* Mathematics *is, in reality, nothing else, but the Doctrine of Proportion; since it only comprehends whatever admits of* Greater *or* Less, *as such.*

5. Fra-

5. Fractions, Vulgar *and* Decimal; (*being only Operations about Proportions*) *where we have bin no less particular, well knowing, that more than an Obscure Conception of these Things is required in him that deserves the Name of an Accomptant. And that the Reader might be compleatly furnish'd with what is necessary to the Understanding the following part of this Treatise, we have given Variety of* Examples *both Numeral and Literal in the several Cases*

6. *The Arithmetic of Surd or Incommensurable Roots is here made plain and easie: Being Operations frequently used in the Practice of Algebra.*

7. *Some general Directions are laid down for the Solution of* Mathematical Problems; *With the Methods of Reducing Equations, Exemplified in all the Cases; As also the Derivation, Composition, and Solution of Equations. And the Method used, as well in the few easie Instances given in* Simple Equations, *as in the Ordinary Rules of* Common Arithmetic (*which fall under that Head*) *if well understood and practis'd, is sufficient to inable the Reader for the Solution of any Question whatever relating to any of these Rules, and that after the most Compendious Manner.*

8. *Arithmetical Progression, with the Solution of the several Cases, and the Operations at Length; that the Learner might be farther acquainted with Analytical Investigations, and the Method of proceeding in the like Calculations. Together with some not unuseful Rules relating to the* Arithmetic *of* Infinites; *by which we have, in its proper place, demonstrated, briefly and directly, some of the most considerable Propositions in* Geometry. *Whence also we have explain'd the Nature and Properties of Figurate Numbers; and shown the Method of Summing up a Series of such Numbers. From these of course are deriv'd the General Theorems for Extracting the Root of any* Binomial *or* Infinitinomial *Power, or for Raising such a Power; On which we insisted the more, because the Method of Approximations by Infinite Series depends thereon: Besides, many valuable Rules and Contrivances for the more ready Solution of some of the greatest Difficulties in* Mathematics *are but as so many immediate Consequences drawn from hence; and, indeed most of what we call* Late Improvements, *or* New Methods of Investigations, *are little else, than the due Application of these Rules.*

A Remarkable Instance of the Use of this Theorem is given, in the next place, for making those Numbers called Logarithms, *with several Examples, and such Explications of the Nature of them, as may render their Construction and Use intelligible to ordinary Capacities. Other Eminent Theorems drawn from hence, are those for Extracting the Roots of* Finite *and* Infinite Equations; *The former we indeavor'd to explain from the successive curious Improvements of Mr.* Raphson, *Mr.* Halley, *and Mr.* Sharp, *with such Illustrations and easie Directions, as might compleat the Learner in the Management of Natural Numbers, by the most expeditious and ready Methods.*

9. *Increasing and Decreasing Geometric Progressions, with the Solution of the several Cases therein, and the Operations at Length.*

10. *Interest, Simple and Compound, solv'd in all its Cases.*

11. *Combinations, Elections, Permutations, and Compositions of Quantities fully handed.*

12. *The Rudiments of* Geometry, *wherein the most valuable and useful Propositions in* Euclid, Archimedes, Apollonius, *and others, are demonstrated in a Natural, Concise, and Easie Method; The principal Properties of (those Figures called) the* Conic-Sections *are consider'd both with, and without respect to the Cone. Together with Rules for the Mensuration of* Lengths, Surfaces, *and* Solids, *investigated by the common general Methods; with the Application of some of those Rules to* Cask-Gauging: *In the Instances there given, some Quantities (as the Learned Dr.* Wallis *look'd upon 'em) are determin'd by Summing up their Elements, by the* Arithmetic of Infinites; *Some (as the Incomparable Sir* Is. Newton *consider'd 'em) by the Velocities of the Motion or Increments, by which they are Generated; according as they render'd the matter either more short, easie, or general: And, for a further Illustration, some by both these Methods.*

13. *The Principles of Projection, containing the Fundamental Rules for the Practice of* Perspective, *or for representing any Object as it appears to the Eye in any given Situation: with the Laws of the* Orthographic, Stereographic, *and* Gnomonic *Projection of the Sphere, by which any one may readily Project the same upon the Plane of any great Circle, or the several Cases of Spheric Triangles, and measure any Arc or Angle when Projected; As also Delineate any Dyall, and with Ease describe the Parallels of the Sun's*

Sun's Declination, or any other Furniture, on the Plane thereof.

14. Trigonometry *both* Plane *and* Spherical, *with the Rules necessary for the Solution of any Case therein, briefly demonstrated.*

15. *The Principles of* Mechanics, *with the general Laws of Motion and their Application in explaining the Powers of Simple Machines and Engines; With some Select Theorems from* Galileus, *Sir* If. Newton, *and* Hugenius, *relating to the Motion of Pendulums, Centripetal Forces, Centres of Gravity,* &c. *To which is added the Doctrine of the Motion of* Projects, *particularly applied to* Gunnery *and Throwing of Bombs; with Directions how to lay a Gun or Mortar to pass so as to strike a Mark with the greatest Certainty and Advantage: where we indeavor'd to follow the Steps of the Learned Geometer Mr.* Halley. *As also the Common Principles of* Optics, *wherein the chief Properties of Refracted and Reflected Rays are briefly deliver'd; with that useful and general Rule, given by the last mention'd Excellent Person, for the* Principal Foci *of Refracted, or Reflected Rays, on any Figur'd* Surface.

The whole is perform'd with as much Perspicuity and Plainness, as the Subject and our Limits wou'd admit; And tho' we have not bin over nice in ranging the Particulars of this Treatise, yet we carefully observ'd the Method used by the most Eminent Mathematicians, who in their Writings were pleased to condescend to the Capacities of Beginners, as, more especially, in the Arithmetical *part,* Vieta, Oughtred, Tacquet, *and* Wallis. *Nor have we bin less wanting in consulting the Works of the most Celebrated Ancient and Modern* Geometers, *that thereby no considerable and useful Proposition or Observation already publish'd, might escape our Notice. We have every where indeavor'd to express things after the clearest and most intelligible manner; and at the same time have avoided all unnecessary Curiosities that might Cloy the Fancy, or Burden the Memory We likewise shun'd the tedious Pomp of Linear Demonstrations, affected by some, in things purely* Arithmetical, *(as*

 those,

those, which depend upon Proportion and the Common Affections of Quantities in general, are reckon'd to be;) since Analytical Demonstrations are not only more General and Abstract, and therefore more Universally applicable to Particular Occasions, but also more Plain and Simple and altogether as Scientific, as those made by Lines and Figures: Nor is there any Difficulty from such to disguise the matter so as to make it look more Geometrical. *And since the Excellency of that Science, which requires Attention, is its being short and plain; we therefore have in every thing studied Brevity with Perspicuity: So that we doubt not, but a Learner, that does not want the necessary Qualifications of Diligence and Industry, will find, with Advantage, That the whole is (as design'd)* A Compendium of the most Select and Primary Principles of Mathematics, *and may serve, at least, as an useful* Introduction *to further Enquiries.*

Thus having given our Impartial Reader a Brief Account of this Treatise, we hope his Candor will prompt him favorably to excuse what is amiss, and amend those Errors which unavoidably attend Things of this Nature.

'Twou'd be here needless to expatiate on the Usefulness of Mathematics, *a Part of Humane Literature, to which all the Concerns of Humane Life are deeply engag'd; Our Pleasure, Security, and Commerce are almost entirely procur'd, maintain'd and improv'd, by the Means of* Civil, Military, *and* Naval Architecture; *and in these there are Variety of illustrious Instances and surprizingly magnificent Pieces of Art, wherein the Effects of* Geometry *appear in an extraordinary manner. And since we are destitute of Senses acute enough to discover the exact Bulk, Motion, and Figure of Bodies, on which their Properties depend, being conceal'd from us, either by their Remoteness, Minuteness, or Transparency: Since therefore, this bounds our Ambitious Desires of viewing the more secret Works of Nature, and our Endeavors to subject 'em to the Doctrine of Magnitude and Numbers, whereby we might, beyond the possibility of Opposition or Doubt, Account for the several* Phænomena *concern'd in our Enquiries; Yet,*

The PREFACE.

Est quodam prodire tenus, si non datur ultra;

and it's somewhat of Content to the Inquisitive Mind (who is ever the least satisfi'd with that, whose Cause is most conceal'd) that by Mechanics *and* Optics, *the Natural Abilities are so far Assisted and Improv'd, as to discover Immensely distant, or Extremely small Objects; Hence we are put in a Capacity of making more Correct Estimations, and of forming juster Notions of the Magnitude, Revolutions and Distances of those Stupendious Fabrics of Nature, which are the constant Subject of our Observations.*

ER-

ERRATA.

PAge 18. l. 18. r. 24. p. 26, l. 21. r. 648. p. 39. l. 6. *ee.* p. 43. l. 12. $x^2 - 16$). p. 44. l. 14. *&c.* p. 47. l. 24. $2ae^2$. p. 53. l. 7. (2 =. p. 54. l. 9. $3ae^2$. p. 60. l. 8. $a + 6d$. ibid. l. 27. dele *which.* p. 64. l. 6. 3^d. p. 68. l. 8. $a^{\frac{n}{r}}$. p. 86. l. 20. r. 384. p. 94. l. 12. *And* $\frac{3}{4}$. p. 95. l. 12. r. $\frac{2}{3}$. ib. l. 15. *the Denominators.* p. 96. l. 13. $x^3 + 30x^2$. p. 175. l. 24. $n\dot{x}$. p. 180. l. 21. dele *the.* p. 185. l. 5. = L, 22. p. 206. l. 6. *Solution.* p. 211. l. 19. r. *as we.* p. 239. l. 25. $-b^2$. p. 244. l. 32. *of* $\overline{A - a}$. p. 249. l. 12. : y^2. ibid. l. 19. r^2 : : 1. p. 276. l. 30. *Declination.*

The Reader is desir'd to correct these; and to excuse any other that possibly may have escap'd Notice.

Synopsis

SYNOPSIS
Palmariorum Matheseos.

General Definitions.

I.

THE *most part of the Objects of our Knowlege may be considered as Capable of* Augmentation *and* Diminution; *and our* Idea *of* Things *as far as they have that Capacity, is what we call* Quantity: *By which Word may be comprehended whatever can be properly said to have* Parts.

SCHOLIUM 1.

Under this Definition of *Quantity*, we may rank *Extension*, *Number*, *Weight*, *Motion*, *Time*, &c. The one being taken as *Greater* or *Less*, *Heavier* or *Lighter*, *Swifter* or *Slower*, &c. in relation to another of the same kind.

For *Great* and *Small*, &c. are only comparative Terms of *Things* that are *Homogeneous*.

SCHOLIUM 2.

And since the Primary and most considerable Property of *Quantity* is a being Capable of *More* or *Less*; Therefore Quantities may be *Added* to, *Subtracted* from, *Multiplied* by one another, and *Divided* into the Parts they contain.

II.

The mutual Relation *of two* Things *of the same kind* Compared *together, in respect of* Quantity, *is call'd* Ratio; *And the* Similitude *of* Ratio's *is call'd* Proportion.

III.

The Knowlege of these Comparisons *of* Quantities, *or the* Relations *they have one to another, is what is generally call'd* **Mathematicks.**

IV.

All Quantities *have their* Parts *either* Continuous *or* Discrete, *that is, either* United *or* Separated.

And that Quantity, *which has its* Parts Separated, *is call'd* Multitude, *and is the Subject of* **Arithmetick**; *But that* Quantity, *which has its* Parts United, *is called* Magnitude, *and is the Subject of* **Geometry.**

SCHO-

SCHOLIUM 1.

The word *Part* only denotes our manner of conceiving a *Thing* when consider'd in Relation to its *Whole*, and may be taken as an Indivisible (Component) of it; or as that whose further Divisibility is not then enquired into. Hence,

SCHOLIUM 2.

Unit is properly a Name given to any *Quantity*, consider'd as an *Indivisible*: and,

Number is a Collection of *Units*; when those *Units* are look'd upon as *Whole* the *Number* goes under the Name of an *Integer*; but if they are taken as *Parts* of a *Whole*, then it is called a *Fraction*.

The Explication of the Signs and Characters used in this Treatise.

=	Signifies	*Equality*, or equal to.
>		*Majority*, or greater than.
<		*Minority*, or less than.
+		*More*, or to be added.
−		*Less*, or to be subtracted.
∽		The *Difference* or *Excess*.
×		*Multiplied* by.
√		*Radicality*.
∺		*Continual Proportion*.

Other Signs or Abbreviations of Words that occur, are explain'd in their own Places.

SYNOPSIS
Palmariorum Matheseos.
PART I.
Containing the
PRINCIPLES
OF
NUMERAL and LITERAL
ARITHMETICK.

SECTION I.
Of Numeral Integers.

CHAP. I.
Notation of Numeral Integers.

DEFINITION I.

NUmeral Notation *teaches how to express in* {*Characters* / *Words*} *any Number propos'd in* {*Words* / *Characters.*}

SCHO-

SCHOLIUM.

1. The *Characters* used to express *Numbers* by, are either

The Ten Numeral *Figures* of the *Arabians*:		Or	The Seven Numeral Letters of the *Romans*.	
Marks	Names		Marks	Names
1	*One*		I	*One*
2	*Two*		V	*Five*
3	*Three*		X	*Ten*
4	*Four*		L	*Fifty*
5	*Five*		C	*Hundred*
6	*Six*		D	*Five Hundred*
7	*Seven*		M	*Thousand*
8	*Eight*			
9	*Nine*			
0	*Nothing* or a *Cypher*.			

2. Each of which *Figures*, besides their own single *Value*, receives several Denominations, according to their *Place* and *Order*.

3. And a *Number* has so many *Places*, as there are *Figures* in it; as, 36487 is a *Number* of five *Places*.

4. The *Order* in whole Numbers, is from the *Right* to the *Left*.

The *Value* of *Places* increases in a *Decuple Proportion*; for every *Place* to the *Left*, is Ten times the *Value* of the next *Place* to the *Right*.

6. Each *Place* also has its *Name*; and those *Names*, for the more easie reading of large *Numbers*, are distinguish'd by *Periods*, *half Periods*, &c.

For as a *Place* / So a *half Period* / And a *Period* } is { *Ten* / *Thousand* / *Million* } times the *Value* of that before it.

7. A *Cypher* is of its self insignificant; but by its *Place* alters the *Value* of the Subsequent Figure.

8. And

8. And since the *Value* of each *Place* is Ten times the *Value* of the next before it, 'tis certain,

That	1 2 3 &c.	in the first Place is	1 2 3 &c.	in the 2d.	10 20 30 &c.	in the 3d.	100 200 300 &c.	&c.	

The Order and Names of Periods, &c.

Periods	Half Periods	Degr.	Places	Order	
Units	un	u	1	1	1
		x	2		
		c	3		
	th	u	4	2	
		x	5		
		c	6		
Millions	un	u	7	3	2
		x	8		
		c	9		
	th	u	10	4	
		x	11		
		c	12		
Billions	un	u	13	5	3
		x	14		
		c	15		
	th	u	16	6	
		x	17		
		c	18		
Trillions	un	u	19	7	4
		x	20		
		c	21		
	th	u	22	8	
		x	23		
		c	24		
		&c.			

COROL.

COROLLARY.

Whence the *Value* of each *Figure* in any *Rank* of *Numbers*, how large soever, is readily found by the following

RULE.

Begin at Units, set a Point under the 7th Place, then, reckoning that as one, count forwards, and set another under the next 7th Place, so continue to the end.

Then the	1st. 2d. 3d. 4th. *&c.*	*Point from Units stands under*	*Millions* *Billions* *Trillions* *Quadrillions* *&c.*

As is evident from the following

EXAMPLE.

Periods	Quadrill.		Trillions		Billions		Millions		Units	
Half Per.	th	un	th	un	th	un	th	un	th	un
Degrees	c x u	c x u	c x u	c x u	c x u	c x u	c x u	c x u	c x u	c x u
Figures.	1 2 3	4 5 6	7 8 9	0 9 8	7 6 5	4 3 2	1 0 1	2 3 4	5 6 7	8 9 1

By this means you have as clear a Notion of, and may as easily read a *Number* of 70 *Places*, as that of seven.

CHAP.

CHAP. II.

Addition of Integers.

DEFINITION.

ADDITION *is the Collection of several* Numbers *or* Quantities *into one* Sum.

PROBLEM 1.

To add Integers *of like Name into one* Sum.

RULE.

Place all Numbers of like kind under one another, Add up the Units, and if their Sum be under Ten, set that Sum underneath;

but if { *equal to* / *above* } *Ten, or Tens, set* { *a Cypher* / *the Excess* } *underneath:*

And for every Ten, carry an Unit to the next Place, and so proceed.

EXAMPLE 1.

To	235	*i. e.*	200 + 30 + 5
Add	432		400 + 30 + 2
Sum	667	=	600 + 60 + 7

The manner of Operation.

2 and 5 = 7	which set under	Units
3 and 3 = 6		Tens
4 and 2 = 6		Hundreds

	235 432		235 432	
For there are	600 60 7	or thus	7 60 600	in the given Numbers.
Therefore	667	=	667	must be their Sum.

EXAMPLE. 2.

To 6985
Add 7645
And 4310

Sum 18940

The manner of Operation.

0 and 5 and 5 = 10	set down	0	and carry 1 to the next place.
1 and 1 and 4 and 8 = 14		4	
1 and 3 and 6 and 9 = 19		9	
1 and 4 and 7 and 6 = 18		18	

6985
7645
4310

For there are	10 Units 13 . Tens 18 .. Hundreds 17 ... Thousands	in the Numbers given.

Therefore 18940 must be the Number *required*, equal to all those *given*.

Since *those things which are equal amongst themselves, are also equal to one another.*

E X-

EXAMPLE III.

To	38796	38796	38796	38796
Add	4638	4638	4638	4638
	30000	14	13	43434
	12000	120	1212	
	1300	1390	3 14	
	120	12000	43434	
	14	30000		
	43434	43434		

PROBLEM II.

To Add Integers of different Names.

RULE.

Place them feverally in the fame Line, with the Sign + *between them, or underftood to be fo.*

EXAMPLES.

1		2	3
To	40	8 *h*	16 *s.*
Add	6	35 *m.*	9 *d.*
Sum =	40 + 6	8 *h* + 35 *m.*	16 *s.* + 9 *d.*
i. e.	46.	8 *h.* 35 *m.*	16 *s.* 9 *d.*

PROBLEM III.

To Add Integers and Parts.

EXAMPLE I.

To	5 F.	16 I.	or	6 F.	4 I.
Add	6.	14.		7.	2.
Sum	11.	30.	=	13.	6.

EXAMPLE II.

To	164 *l.*	15 *s.*	6 *d.*	or	164 *l.*	15 *s.*	6 *d.*
Add	79.	12.	9.		79.	12.	9.
Sum	244.	8.	3.	=	243.	27.	15.

CHAP. III.

Subtraction of Integers.

DEFINITION.

SUBTRACTION *is the taking of one Number or Quantity from another, to find their* Difference.

SCHOLIUM.

The {greateſt / leaſt} *of the given Numbers is called the* {Minuend / Subducend}. *And the Number found is call'd the* Remainder *or* Difference.

PROB.

PROBL. I.

To Subtract Integers of like Names, when the Superiour Numbers are greater than, or equal to their Inferiours,

RULE.

1. *Place the Subducend under the Minuend, and draw a Line under both.*
2. *Begin at the Right-hand, take the less from the greater, or Equals from Equals, and set the Difference of each Row underneath.*

Example, *In Integers alone.*

Minuend 638
Subducend 213

Remainder 425

The manner of Operation.

3, 1, 2 from 8, 3, 6 and there remain 5, 2, 4 which set below,

That is,

	3		8		5
	10		30		20
	200	from	600	remainder is	400
Theref.	213		638		425

For since *the Whole is equal to the Sum of all its Parts, therefore the Subduction of all the Parts is the same with the Subduction of the whole.*

Examples,

Examples, *In Integers and Parts.*

Minuend	27 *d.*	48*l.*	36*ll.*	146 *l.*	18 *s.*	6 *d.*
Subducend	12.	31.	24.	22.	8.	2.
Remainder	15.	17.	12.	124.	10.	4.

PROB. II.

To Subduct Integers of the same Name, when some of the Superiour Numbers are less than their Inferiour.

RULE.

1. *Place your Numbers, and begin as before.*
2. *And according to their respective Value, take one of the next Denomination, out of which Subduct; and to the Remainder, add the Superiour, setting their Sum underneath.*
3. *Then add what you took to the next place on the Left-hand; and so proceed by this, or the former Rule.*

Example, *In Integers alone.*

From	2537
Subd.	1648
Rem.	889

The manner of Operation.

	8		17		9	
1 and 4 =	5	from	13	Rem.	8	which set below
1 and 6 =	7		15		8	

That

That is,

8	from	7 + 10	Rem.	9
10 + 40		30 + 100		80
100 + 600		500 + 1000		800
1000 + 1000		2000 + 0000		0000
Theref. 1648		2537		889

For by saying 8 from 17, I add Ten to the *Minuend*, but I add also the same to the *Subducend*, by saying 1 and 4 = 5; therefore the Remainder must be the same.

The Operation also may be thus;

8	from	17	Rem.	9 Units
4		13—1		8 Tens
6		15—1		8 Hundreds
1		2—1		0.

That is,

8	from	7 + 10	Rem.	9
40		30 + 100 — 10		80
600		500 + 1000 — 100		800
1000		2000 + 0000 — 1000		0000
Theref. 1648		2537		889

For by adding a Ten to the Units, and taking it away from the Tens, the value of the Number is not changed.

Examples, *In Integers and Parts.*

From	5 *s.*	3 *d.*	*i. e.*	5 *s.*	15 *d.*	or	4 *s.*	15 *d.*
Subd.	2.	9.		3.	9.		2.	9.
Rem.	2.	6.	=	2.	6.	=	2.	6.

From

From.	246 *l.*	3 *s.*	4 *d.*	*i. e.*	246 *l.*	23 *s.*	16 *d.*
Subd.	68.	10.	6.		69.	11.	6.
Rem.	177.	12.	10.	=	177.	12.	10.

or

From	245 *l.*	22 *s.*	16 *d.*
Subd.	68.	10.	6.
Rem.	177.	12.	10.

THEOREM.

In Subtraction, the Subducend together with the Remainder, is equal to the Minuend.

For all the Parts taken together, are equal to the whole.

And if the Subducend be taken from the Minuend, there rests the Remainder.

But if a Part be taken from the Whole, the Remainder will be the other Part.

Therefore the Subducend, together with the Remainder are all the Parts of the Minuend, and consequently equal to it.

COROLLARY.

Hence, *Addition* and *Subduction*, serve Reciprocally to prove each other.

For Addition and Subduction are opposite in all Cases; and what is done by the one, is undone by the other.

Thus

Thus if to	6	And if from	10
be added	4	Subduct.	4
Sum	10	Rem.	6

That is, if 6 + 4 = 10, then 10 − 4 = 6.

CHAP. IV.

Multiplication of Integers.

DEFINITION.

MULTIPLICATION *is a manifold* Addition, *or the repeating a given* Quantity, *as often as required; That is, to take a* Quantity *so many* Times, Part *or* Parts *of a* Time, *as is represented by another.*

SCHOLIUM.

The Number { to be Repeated / of Repetitions } is called the { *Multiplicand* / *Multiplier.* }

The Sum of the Number so often Repeated, is called the *Product*. Both *Multiplicand* and *Multiplier* are call'd *Factors.*

COROLLARY.

Hence, As Unit, is to one Factor:
So is the other Factor, to the Product.

CASE. I.

To Multiply single Numbers by one another,

EXAMPLE.

Mult. 4
by 3

Prod. 12 *i. e.* $4+4+4=4\times3=3\times4.$

SCHOLIUM.

All the variety that can happen in this Case, is express'd in the following

Table of MULTIPLICATION.

1	2							
2	4	3						
3	6	9	4					
4	8	12	16	5				
5	10	15	20	25	6			
6	12	18	24	30	36	7		
7	14	21	28	35	42	49	8	
8	16	22	32	40	48	56	64	9
9	18	27	36	45	54	63	72	81

LEM-

LEMMA. I.

The Product of any two Numbers, is equal to the several Products made by multiplying one of those Numbers by the several Parts of the other.

Thus $\left.\begin{matrix}9\\3\end{matrix}\right\}$ *i. e.* $\left\{\begin{matrix}4+3+2\\3\end{matrix}\right.$ $\left.\begin{matrix}34\\2\end{matrix}\right\}$ *i. e.* $\left\{\begin{matrix}30+4\\2\end{matrix}\right.$

$27 = 12+9+6 \quad 68 = 60+8$

For, since the *Whole*, and all its *Parts* taken together, make but one and the same thing:

Therefore the *Multiplying* the *Whole*, and all its *Parts*, gives the same *Product*.

CASE II.

To Multiply a Compound Number by a Single one,

EXAMPLE.

Mult. by $\left.\begin{matrix}6452\\3\end{matrix}\right\}$ or $\left\{\begin{matrix}6000+400+50+2\\3\end{matrix}\right.$

Product. $19356 = 18000+1200+150+6$

That is $\left.\begin{matrix}6452\\3\end{matrix}\right\}$ or $\left\{\begin{matrix}6452\\3\end{matrix}\right\}$ or $\left\{\begin{matrix}6452\\3\end{matrix}\right.$

11156	6	18000
82	150	1200
	1200	150
19356	18000	6
	19356	19356

$$\text{For since 3 times}\left\{\begin{array}{r}6000\\400\\50\\2\end{array}\right\}\text{ is }\left\{\begin{array}{r}18000\\1200\\150\\6\end{array}\right\}$$

$$\text{Therefore 3 times } 6452 \text{ is } 19356$$

LEMMA II.

The Product of any two Quantities, is equal to the several Products made by Multiplying all the Parts of the one, by all the Parts of the other.

$$\text{Thus}\left\{\begin{array}{r}12\\7\end{array}\right\}\text{or}\left\{\begin{array}{r}10+2\\3+4\end{array}\right\}\text{or}\left\{\begin{array}{r}6+4+2\\3+4\end{array}\right.$$

$$84=30+6+40+8=18+12+6+24+16+8$$

CASE III.

To Multiply one Compound Number by another.

RULE

1. *Place each Number respectively under its kind.*
2. *Multiply each Figure of the Multiplicand, by each Figure of the Multiplier; and observe to set the first Figure of each respective Product under that Figure of the Multiplier by which it was made.*
3. *Add the several Products together for the whole Product.*

EXAMPLE I.

$$\begin{array}{lr}\text{Mult.} & 123\\ \text{by} & 23\\ \hline & 369\\ & 246\\ \hline & 2829\end{array}$$

$$\left\{\begin{array}{r}123\\20\end{array}\right\}+\left\{\begin{array}{r}123\\3\end{array}\right\}\text{or}\left\{\begin{array}{r}100+20+3\\20+3\end{array}\right.$$

$$2460+369=2000+400+60+300+60+9$$

For

For:

3 ×	100	=	300
	20	=	60
	3	=	9
20 ×	100	=	2000
	20	=	400
	3	=	60
Therefore 23 ×	123	=	2829

EXAMPLE 2.

Multiplicand	5326
Multiplier	427
	37282
	10652
	21304
Product	2274202

The manner of Operation.

I. 7 times { 6 = 42; 2, + 4 = 18; 3, + 1 = 22; 5, + 2 = 37 } subscribe { 2; 8; 2; 37 } carry { 4; 1; 2 }

II. 2 times { 6, = 12; 2, + 1 = 5; 3, = 6; 5, = 10 } subscribe { 2 and carry 1; 5; 6; 10 }

III. 4 times { 6, = 24; 2, + 2 = 10 } subscribe { 4; 0 } carry { 2; 1 }
&c.

Multiplicand	5326
Multiplier	427

For since

7 ×	6 =	42	1st.	Product.
	20 =	140	2	
	300 =	2100	3	
	5000 =	35000	4	
20 ×	6 =	120	5	
	20 =	400	6	
	300 =	6000	7	
	5000 =	100000	8	
400 ×	6 =	2400	9	
	20 =	8000	10	
	300 =	120000	11	
	5000 =	2000000	12	

Therefore 2274202 is the Sum of the *Products* of all the *Parts* of the *Multiplicand* 5326, Multiplied by all the *Parts* of the *Multiplier* 427, and consequently equal to 5326 × 427, by *Lemma* 2.

SCHOLIUM I.

When either the Multiplicand, Multiplier, or both, have Cyphers towards the Right-hand; then *Multiply the Significant Figures by the former Rules, and annex to the Product as many Cyphers as there are in the Multiplicand and Multiplier.*

EXAMPLES.

246 by	10	gives	2460
	100		24600
	1000		246000
	10000		2460000

143	9520	624000
20	3400	4300
2860	3808	1872
	2856	2496
	32368000	2683200000

SCHOLIUM 2.

When there are Cyphers in the Multiplier; *then Subscribe those Cyphers in Order, before the particular Product of the next Multiplier by the Multiplicand.*

EXAMPLE.

427	6700042
306	100005
2562	33500210
12810	670004200000
130662	670037700210

SCHOLIUM 3.

To Multiply by any Compound Number under 20; you need only,

Set the Product made by the Unit Figure of the Multiplier, a Place further to the Right-hand, and add thereto the Multiplicand. Thus,

Multipli-

Multiplicand	25436
Multiplier	14
	101744 1ft. Product
Product	356104

Or shorter, thus.

```
25436
   14
------
356104
```

The manner of Operation.

			Subscr.	carry
4 by	6 = 24		4	2
	3 = 12,	+ 2 + 6 = 20	0	2
	4 = 16,	+ 2 + 3 = 21	1	2
	5 = 20,	+ 2 + 4 = 26	6	2
	2 = 8,	+ 2 + 5 = 15	5	1
	Then	2 + 1 = 3	3	

SCHOLIUM 4.

If a Quantity be Multiplied by the Component Parts of the Multiplier, the Product will be the same as if it had been Multiplied by the Multiplier it self.

Thus, 245 by 7, and the Product by 6, is the same as if 245 was Multiplied by 7×6, that is, by 42.

CHAP.

CHAP. V.

Division of Integers.

DEFINITION.

DIVISION *is a Manifold* Subduction; *or the taking of one* Number *or* Quantity *out of another, as often as possible.*

As 6 Divided by 2 gives 3.
For 6 — 2 — 2 — 2 = 0.

SCHOLIUM.

The Number { to be *Divided* / *Dividing* } is called { *Dividend* / *Divisor* }

And the Number of *Times* they contain each other, is called *Quotient.*

COROLLARY.

Hence, as the Divisor, is to the Dividend: So is Unit, to the Quotient.

The *Terms* in *Division* are thus Placed.

Divisor Dividend Quotient
2) 6 (3

Or,

Dividend 6 / *Divisor* 2 (3 *Quotient.*

E PROB.

PROBLEM.

To Divide one Number by another.

RULE.

1. *Set a Point under the last of the Left-hand Places in the Dividend, out of which the Divisor may be taken; and the Number of Places to the Right of that Point inclusive, gives the Number of Places in the Quotient.*
2. *Try how often you can take the Divisor out of that first part of the Dividend, setting the Number of Times in the Quotient; then Multiply the Divisor thereby, Subduct the Product out of the said part of the Dividend, and subscribe the Remainder.*
3. *To the Right of the Remainder, set the next Figure of the Dividend, from which take the Divisor as often as you can, setting the Number of Times in the Quotient, Multiply the Divisor thereby, Subducting the Product as before; and in this manner the Operation must be repeated, till you come to the end.*

EXAMPLE I.

Divisor	*Dividend*	*Quotient*
2)	642	(324
	···	
	6··	
	04·	
	4·	
	02	
	2	
	0	

For

$$\text{For 2 in}\left\{\begin{array}{r}600\\40\\8\end{array}\begin{array}{l}(300\\(\ 20\\(\ \ 4\end{array}\right\}\text{times}$$

Therefore 2) 648 (324 times

EXAMPLE. II.

```
3) 19358 (6452
   ....
   18...
   -----
    13..
    12..
    ----
     15.
     15.
     ---
      08
       6
      --
Rem.   2
```

$$\text{For 3 in}\left\{\begin{array}{r}18000\\1200\\150\\8\end{array}\begin{array}{l}(6000\\(\ 400\\(\ \ 50\\(\ \ \ 2\end{array}\right\}\text{times}$$

Rem. 2

Therefore 3) 19358 $(6452\frac{2}{3}$

EXAM-

EXAMPLE III.

```
Divisor Dividend Quotient
  24)    5664    (236
          ...
          48
         ----
          86
          72
         ----
          144
          144
         ----
          000
```

The manner of Operation.

1. Having placed the Numbers, and pointed them as the Rule Directs: and finding I can have 24 in 56 but 2 times, therefore I set 2 in the *Quotient*, the *Divisor* Multiplied thereby, gives 48, which Subducted from 56 leaves 8; to the Right of 8 I set the next Figure of the *Dividend*, 6.

2. I can have 24 in 86 thrice, therefore I set 3 in the *Quotient*; then thrice 24 is 72, which Subducted from 86 leaves 14; to the Right of 14, I set 4 the next and last Figure of the *Dividend*.

3. I can have 24 in 144, 6 times, therefore I set 6 in the *Quotient*; then 6 times 24 is 144, which *Subducted* from 144, leaves nothing. Hence, I conclude; That 24 is contained 236 times in 5664.

The

The *Reason* of this Operation is evident from the following Process.

```
                5664
I.      24 in) 5600 (200 times
     200 × 24 = 4800
                ————
                 800
             +    60
                ————
II.     24 in)   860 (30 times
      30 × 24 =  720
                ————
                 140
             +     4
                ————
III.    24 in)   144 (6 times
       6 × 24 =  144
                ————
                 000
```

And since	24) 4800	(200	times
	24) 720	(30	
	24) 144	(6	
Therefore	24) 5664	(236	

SCHOLIUM 1.

If the *Divisor* be greater than the *Dividend*; or if they be of different Names, then, *The Dividend set above, and the Divisor below a Line, represent the Quotient.*

Ex. I. 4 Divided by 9, the Quotient is $\frac{4}{9}$

II. 24 Feet Divided by 6 Yards.

The Quotient is $\frac{24 \text{ Feet}}{6 \text{ Yards}}$.

SCHOLIUM. 2.

If the *Divisor* have *Cyphers* towards the Right Hand; then,

Cut off so many of the Right Hand Places of the Dividend as there are Cyphers in the Divisor, which annex to the Remainder, when the Operation is finished.

EXAMPLES.

I. 10) 267|8 $(267\frac{8}{10}$

II. 300) 69|52 $(23\frac{52}{300}$

III. 3200) 2457|26 $(76\frac{2526}{3200}$

```
        ..
       224
       ———
        217
        192
        ———
         25
```

THEOREM.

The Divisor Multiplied by the Quotient is equal to the Dividend.

As suppose 15 was Divided by 3, the *Quotient* will be 5.

But the *Quotient* 5 is so many Parts of the *Dividend* 15, *as Unit is of the Divisor* 3, *that is, the third Part*; *by Definit. of Division.*

Therefore, If the *Quotient* be taken so many times as there are Units in the *Divisor* 3, the Sum will be equal to the *Dividend* 15.

And all the Parts (5 times 3) taken together are equal to the whole (15).

Therefore, The *Product* of the *Divisor* by the *Quotient* is equal to the *Dividend.*

COROL-

COROLLARY.

Hence, 'tis evident, That *Multiplication* and *Division* serve Reciprocally to prove each other.

For *Multiplication* and *Division* are two contrary Operations, and what is done by the one, is undone by the other.

Thus, if 5 *Multiplied* by 3, give 15.
Then 5 is contain'd 3 times in 15.
Therefore $\frac{15}{3}$ (= 5 and $\frac{15}{5}$ (= 3.
Also if 15 Divided by 3 give 5,
Then 5 taken 3 times is 15.
Therefore $5 \times 3 = 15 = 3 \times 5$.

SECT.

SECT. II.

Of Literal *or* Algebraic Integers.

CHAP. I.

Notation of Algebraic Integers.

DEFINITION.

ANY Number or Quantity whatsoever, whether known or unknown, may be universally express'd by Notes, Characters *or* Letters *of the Alphabet, at Pleasure: And this way of* Notation, *applied to* Arithmetical *Operations, is what is commonly call'd* ALGEBRAICAL *or* LITERAL ARITHMETIC.

SCHOLIUM.

1. In Operations perform'd by *Literal Arithmetic*, all *Quantities*, *known* or *unknown*, are represented by *Letters*, with prefix'd *Marks* or *Signs*, whereby, (according to the Nature of the *Proposition*) they may be so order'd by *Addition*, *Subduction*, *Multiplication*, *Division*, &c. as if each particular Part was actually known; so that the

the Position and Relation of one Quantity to another is visible thro' the whole Course of the Process, and consequently that of the *known* to the *unknown*.

2. *Quantities* represented by { *the same* / *different* } Letters in any Operation, are supposed to be of { *the same* / *different* } Value.

3. And between Quantities of *different values*, there may be an *Equality*, as $6a = 2b$; 6 Feet = 2 Yards.

4. The number of Times any *Quantity* is taken, must be prefix'd to it, and is call'd a *Coefficient*, or *Co-factor*.

5. A Quantity without a { *Coefficient* / *Sign* } is supposed to have { *Unit* / *the Sign* + } prefix'd to it.

6. { *Simple* / *Compound* } Quantities, are those which { have but one Member. / are connected by + and −. }

And since *Addition*, *Subduction*, *Multiplication* and *Division*, are the Common Affections of all *Quantities*, therefore we shall in the next place endeavour with all the Brevity and Plainness possible, to apply these Rules to *Letters*, as we have to *Numbers*.

AXIOMS.

1. If *to* or *from* equal Quantities, equal ones be *Added* or *Subducted*, their *Sum* or *Remainder* will be equal.

2. If equal Quantities be *Multiplied* or *Divided* by equal ones, their *Products* or *Quotients* will be equal.

CHAP. II.

Addition of Algebraic Integers.

CASE. I.

When Quantities have the same Name, *and the same* Sign,

RULE.

Let the propos'd Quantities be Collected, and their respective Signs adjoyn'd.

EXAMPLES.

To $+a$	$+2b$	$-2n$	$yy+2x-2$
Add $+a$	$+5b$	$-n$	$yy+3x-5$
Sum $+a+a$	$2b+5b$	$-3n$	$2yy+5x-7$
Or $+2a$	$+7b$		

For 'tis manifest from the Common way of Numbering, that $2+5=7$, of any thing of like Name; as 2 Miles and 5 Miles are 7 Miles.

CASE II.

When Quantities have the same Name, *but different* Signs,

RULE.

Let them be Subducted from each other, and the Sign of the Greater adjoyn'd to the Remainder.

EX-

EXAMPLES.

To	$+3a$	$-3x$	$+4z$	$6aa-x+30$
Add	$-2a$	$+2x$	$-4z$	$x-20-6aa$
Sum	$+3a-2a$	$-3x+2x$	$+4z-4z$	$+10$
Or	$+a$	$-x$	0	

For to Add a *Negative*, is to take away a *Positive*; Therefore, to Connect a *Negative* and a *Positive*, is to make the one and destroy the other.

Thus, If *A* has 600 *l.* and owes 400 *l.* 'tis plain that the Sum, or his Worth is but 200 *l.*

And if *A* has 600 *l.* and owes 900 *l.* then the Worth is — 300 *l.* or 300 *l.* worse than nothing.

CASE III.

When Quantities are of different Names,

RULE.

Let them be set down in order, with their own Signs prefix'd.

EXAMPLES.

To	a	$5b$	$a+b$	$3a-2z-3x+6$
Add	c	$4m$	$c-d$	$47+pp+3a+2z$
Sum	$a+c$	$5b+4m$	$a+b+c-d$	$6a+53-3x+pp$

For 4 *Miles* and 5 *Hours*, make neither 9 *Miles* nor 9 *Hours*.

CHAP. III.

Subduction of Algebraic Integers.

Case { 1. + from +; and − from − }
{ 2. − from +; and + from − }
Both perform'd by this

General RULE.

Let (or suppose) all the Signs of the Subducend be changed, Then the Quantities Collected, (as in Addition) give the Remainder.

EXAMPLES In *Case* 1.

From	$+3a$	$+ax$	$-2b$	$5n+4z+6$
Subd.	$+2a$	$+ax$	$-2b$	$6o+3z+9$
Rem.	$+3a-2a$	$ax-ax$	$-2b+2b$	$5n+z-63$
Or	$+a$	o	o	

For to *take away* any *Thing*, is the same as to *Subjoyn* the *Defect* of that *Thing*.

Therefore, to take away the { *Affirmation* / *Negation* } of any thing, is to make it { *Negative* / *Affirmative*. }

EXAMPLES, In *Case* 2.

From	$+3a$	$-3x$	$+6a$	$3aa-5a+7$
Subd.	$-2a$	$+2x$	$-2a-y$	$aa+a-2$
Rem.	$+5a$	$-5x$	$+8a+y$	$2aa-6a+9$

For to *take away* the want of a *Thing*, is to *Add* that very *Thing*, by taking away the Negation of it.

But to *take away* the *Being* or *Affirmation* of a *Thing*, must necessarily produce the *Want* or *Negation* of that *Thing*.

SCHO-

SCHOLIUM.

In *Addition* and *Subduction*, it is indifferent as to Order, how the ſeveral Quantities do ſtand, ſo that each has its own Sign.

For $x-y+z=x+z-y=-y+z+x=z+x-y=n$
Or $8-6+2=8+2-6=-6+2+8=2+8-6=4$

CHAP. IV.

Multiplication of Algebraic Integers.

NOTE.

That in *Multiplication* $\left\{\begin{matrix}\textit{Like}\\ \textit{Unlike}\end{matrix}\right\}$ Signs give $\left\{\begin{matrix}+\\ -\end{matrix}\right\}$ in the Product.

Multiplicand Multiplier Product

That is $\left\{\begin{matrix}\pm\\ \pm\end{matrix}\right\}$ into $\left\{\begin{matrix}\pm\\ \mp\end{matrix}\right\}$ gives $\left\{\begin{matrix}+\\ -\end{matrix}\right.$

For to Multiply an *Affirmative* by $\left\{\begin{matrix}\text{an } \textit{Affirmative}\\ \text{a } \textit{Negative}\end{matrix}\right\}$ Quantity, is ſo often to repeat the $\left\{\begin{matrix}\textit{Affirmation}\\ \textit{Negation}\end{matrix}\right\}$ of it; by *Def.* of *Mult.*

Therefore $\left\{\begin{matrix}+\\ -\end{matrix}\right\}$ into $+$, gives $\left\{\begin{matrix}+\\ -\end{matrix}\right.$

But to Multiply a *Negative* by a *Negative* Quantity, is ſo many times to deny that *Negation.*

And ſo many times to deny the *Negation* of a *Thing*, is ſo many times to *Affirm* that *Thing.*

Therefore $-$ into $-$ gives $+$.

PROB.

PROBL. I.

To Multiply Simple Quantities.

RULE.

Joyn the Factors together, and prefix to them the Product of the Coefficients, if there be any, observing the foregoing Note.

EXAMPLES.

Mult.	$+a$	$-a$	$+a$	$-a$	$+a$	$-3ab$
By	$+b$	$-b$	$-b$	$+b$	$+3$	5
Prod.	$+ab$	$+ab$	$-ab$	$-ab$	$+3a$	$-15ab$

PROB. II.

To Multiply Compound Quantities,

RULE.

Each part of the Multiplier must be drawn into each part of the Multiplicand; by Lem. 2. Chap. 1. §. 1.

Examples, Of Compounds by Simples.

Mult.	$a+z$	$x-y$	$rr-6$	$a-n+m$
By	n	x	3	$-y$
Prod.	$na+nz$	$xx-xy$	$3rr-18$	$-ya+yn-ym$

Examples,

Examples, Of Compounds by Compounds.

$$\begin{array}{lrr}
\text{Mult.} & a+e & \\
\text{By} & a-e & \\
\hline
 & aa+ae & \\
 & -ae-ee & \\
\hline
\text{Prod.} & aa-ae &
\end{array}
\qquad\Bigg|\qquad
\begin{array}{r}
2x-a+3 \\
x-4 \\
\hline
2xx-xa+3x \\
-8x+4a-12 \\
\hline
2xx-xa-5x+4a-12
\end{array}$$

$$\begin{array}{ll}
\text{Mult.} & 4a^3+3a^2-2a+1 \\
\text{By} & a^2-5a+6 \\
\hline
\text{Prod.} & 4a^5-17a^4+7a^3+29a^2-17a+6.
\end{array}$$

$$\begin{array}{llll}
\text{Mult.} & ZZZ & -2nZZ & -nnn \\
 & & +a & +naa \\
 & & & -2n^2a \\
\text{By} & ZZ & +2nZ & +4nn \\
 & & -a & -4na \\
 & & & +aa
\end{array}$$

$$\begin{array}{llllll}
Z^5 & -2nZ^4 & -4n^2Z^3 & -n^3Z^2 & -2n^4Z & -4n^5 \\
 & +a & +2na & +a^3 & +2na^3 & +4n^2a^3 \\
 & & +2na & -2n^2a & -4n^3a & -8n^4a \\
 & +2n & -aa & & +n^3a & +4n^4a \\
 & -a & & -8n^3 & -a^4 & -4na^4 \\
 & & +4n^2 & +4n^2a & +2n^2a^2 & +8n^3a^2 \\
 & & -4na & +8n^2a & & -n^3a^2 \\
 & & +aa & -4na^2 & & +a^5 \\
 & & & -2na^2 & & -2n^2a^3 \\
 & & & +a^3 & & \\
\hline
\text{Pr.}\ Z^5 & * & * & -9n^3ZZ & -2n^4Z & -4n^5 \\
 & & & +2a^3 & +2na^3 & +2n^2a^3 \\
 & & & +10n^2a & -3n^3a & -4n^4a \\
 & & & -6na^2 & -a^4 & -4na^4 \\
 & & & & +2n^2a^2 & +7n^3a^2 \\
 & & & & & +a^5
\end{array}$$

SCHO-

SCHOLIUM 1.

Sometimes *Products* are express'd only by the Quantities to be *Multiplied* with the sign × between them.

Thus, The Product of $a+x$ by $e+z$, is $\overline{a+x} \times \overline{e+z}$. And the Product of $a+x$ by $m-n+y$, and that Product by $e+zr$

$$\text{Is} \quad \overline{\overline{a+x} \times \overline{m-n+y}} \times \overline{e+zr}.$$

SCHOLIUM 2.

The *Quantity* produc'd by the *Multiplication* of *Two*, *Three*, &c. *Quantities*, is said to be of *Two*, *Three*, &c. *Dimensions*; and the *Quantities* thus *Multiplied*, are called *Roots*;

$$\text{Thus} \left\{\begin{array}{l} ab \\ abc \\ \&c. \end{array}\right\} \text{is a Quantity of} \left\{\begin{array}{l} 2 \\ 3 \\ \&c. \end{array}\right\} \textit{Dimensions}, \text{ whose } \textit{Roots} \text{ are} \left\{\begin{array}{l} a, b. \\ a, b, c. \\ \&c. \end{array}\right.$$

But if the *Roots* are the same, then the Quantities produced are usually call'd *Powers*, as

$$\left\{\begin{array}{l} a \\ aa \\ aaa \\ \&c. \end{array}\right\} \text{is the} \left\{\begin{array}{l} \text{1st.} \\ \text{2d.} \\ \text{3d.} \\ \&c. \end{array}\right\} \textit{Power} \text{ of } a, \text{ and may be thus express'd} \left\{\begin{array}{l} a \\ a^2 \\ a^3 \\ \&c. \end{array}\right\}$$

CHAP.

CHAP. V.

Division of Algebraic Integers.

General RULE.

Set the Divisor under the Dividend, with a Line between them.

EXAMPLES.

Divide	a	$nn+r$	$a+bc-dd$	$an-nx$
By	e	$x-y$	$x+zz$	$a+u$
Quotient	$\frac{a}{e}$	$\frac{nn+r}{x-y}$	$\frac{a+bc-dd}{x+zz}$	$\frac{an-nx}{a+u}$

For by *Common Division*, in Dividing 6 by 2, 'tis the same thing whether 3, or $\frac{6}{2}$ be the *Quotient*.

But sometimes this Mark $\div$ is used as a sign of *Division*.

SCHOLIUM 1.

If a *Quantity* is found to be a Common *Multiplier* in both, *it may be expunged from both*; Observing that

Divisor	Dividend	Quotient
$+$	$+$	$+$
$-$	$-$	$+$
$+$	$-$	$-$
$-$	$+$	$-$

From the like *Reason* with that in *Multiplication*.

EXAMPLES.

Divide	$+ae$	$-ae$	$+ae$	$-ae$	$na+ne$
By	$+a$	$-a$	$-a$	$+a$	n
Quot.	$+e$	$+e$	$-e$	$-e$	$a+e$

For $e\times a = ae$, and $\overline{a+e}\times n = na+ne$. by *Theor.* Chap. 5. §. 1.

Divide $\dfrac{am+an}{m+n}$ (*a*. By | $\dfrac{za-ac+zn-cn}{z-c}$ $(a+n$

SCHOLIUM 2.

In reducing these *Quotients* into lower Terms, it happens frequently, when the *Dividend* and *Divisor* are *Multinomial* Quantities, that there may be a Common Multiplier of both, which does not immediately appear; yet is found (as in Ordinary *Arithmetic*) by this

RULE.

Find one Member of the Quotient, by which Multiply the Divisor, Subducting the Product from the Dividend, and by the Remainder seek another Member of the Quotient, and proceed thus, till the Operation be finished.

EXAMPLES.

$x-z)\ x^3-z^3\ (x^2+xz+z^2$

$0+x^2z-z^3$

$0+xz^2-z^3$

$0\qquad 0$

The

The manner of Operation.

1. Find what Quantity Multiplied by x (the firſt Member of the Diviſor) will give x^3, (the firſt Member of the Dividend) and that muſt be x^2; whereby Multiply $x - z$, the Product $x^3 - x^2 z$ Subducted from the Dividend, leaves $x^2 z - z^3$.

2. Seek what Quantity Multiplied by x, gives $x^2 z$, that muſt be zx; which (as before) Multiplied and Subducted leaves $xz^2 - z^3$.

3. In the ſame manner, ſeek the third Member of the Quotient, which Multiplied and Subducted, leaves 0.

$$x - 19)\; x^6 - 8x^4 - 124x^2 - 64\; (x^4 + 8x^2 + 4$$
$$x^6 + 16x^4$$
$$\overline{\quad - 8x^4 \quad}$$
$$+ 8x^4 - 128x^2$$
$$\overline{\quad -4x^2 \quad}$$
$$+4x^2 - 64$$
$$\overline{\quad 0 \quad}$$

$$\begin{array}{l} x^2 - n^2 \\ - y^2 \end{array}\Big)\; \begin{array}{l} x^6 + n^2 x^4 - n^4 x^2 - n^6 \\ \quad\;\; - 2y^2 \;\; + y^4 \quad - 2n^4 y^2 \\ \qquad\qquad\qquad\quad - n^2 y^4 \end{array} \Big(\begin{array}{l} x^4 + 2n^2 x^2 + n^4 \\ \quad - y^2 \quad - n^2 y^2 \end{array}$$

$$\overline{\quad\quad\quad}$$
$$\begin{array}{l} - n^2 x^4 \\ - y^2 \end{array}$$
$$\overline{\quad\quad\quad}$$
$$\begin{array}{l} + 2n^2 x^4 - 2n^4 x^2 \\ - y^2 \qquad - n^2 y^2 \\ \qquad\quad + y^4 \end{array}$$
$$\overline{\quad\quad\quad}$$
$$\begin{array}{l} + n^4 x^2 - n^6 \\ + n^2 y^2 - 2n^4 y^2 \\ \qquad\quad - n^2 y^4 \end{array}$$
$$\overline{\quad\quad\quad}$$
$$0$$

SCHO-

SCHOLIUM 3.

If the *Divisor* be not an even Part of the *Dividend*, the Operation may either be terminated by annexing to the Quotient, the Remainder set over the Divisor, with a Line drawn between them; or else continued on in an *Infinite Series*.

EXAMPLES.

Divide $\dfrac{9n^3y^4}{2ny}\left(=\dfrac{9n^2y^3}{2}\right.$ | $\dfrac{8n^3y^6}{4nya}\left(=\dfrac{2ny^5}{a}\right.$
By

$$2n)\ \underline{8n^6+4n^2a^2y^2}-3n^2a^4\left(4n^5+2na^2y^2-\frac{3na^4}{2}\right.$$

$$\quad 0 \quad 0$$

$$-2n)\ \underline{10n^5-12n^3a^2}-6a^2c^3\left(-5n^4+6n^2a^2+\frac{3a^2c^3}{n}\right.$$

$$\quad 0 \quad 0$$

$$aa-ee)\ aae\ \left(e+\frac{e^3}{aa}+\frac{e^5}{a^4}+\frac{e^7}{a^6}+\frac{e^9}{a^8}+\&c.\right.$$

$$\underline{aae-e^3}$$

$$e^3$$

$$\underline{e^3-\frac{e^5}{aa}}$$

$$\frac{e^5}{aa}$$

$$\underline{\frac{e^5}{aa}-\frac{e^7}{a^4}}$$

$$\frac{e^7}{a^4}$$

&c.

$$aa-zz)\,aaax\,(ax+\frac{z^2x}{a}+\frac{z^4x}{aaa}+\frac{z^6x}{a^5}+\&c.$$

$$aaax-azzx$$

$$azzx$$

$$azzx-\frac{z^4x}{a}$$

$$\frac{z^4x}{a}$$

$$\frac{z^4x}{a}-\frac{z^6x}{a^3}$$

$$\frac{z^6x}{a^3}$$

$$\frac{z^6x}{a^3}-\frac{z^8x}{a^5}$$

$$\frac{z^8x}{a^5}$$

&c.

$$1-x)\,1\,(1+x+xx+x^3+x^4,\ \&c.$$

$$1-x$$

$$x$$

$$x-xx$$

$$xx$$

$$xx-xxx$$

$$xxx$$

$$xxx-x^4$$

$$x^4$$

$$x^4-x^5$$

$$x^5$$

&c.

SCHO-

SCHOLIUM 4.

In *Multiplication* and *Division*, 'tis indifferent (as to Order) how the *Quantities* ſtand, ſo the Operations be Performed ſucceſſively.

CHAP. VI.

Involution of Quantities.

DEFINITION.

A Quantity *Multiplied into its ſelf any Number of Times is ſaid to be* Involved; *the* Products *ariſing are called* Powers, *and the* Quantitiy *ſo Multiplied a* Root.

$$\text{Thus}\begin{cases} a\times a = aa \text{ or } a^2 \\ a\times a\times a = aaa \text{ or } a^3 \end{cases}$$
&c.

Here a is the *Root* or *Firſt Power*; a^2, a^3, &c. The *Second*, *Third*, &c. *Powers* of it.

SCHOLIUM.

Some of theſe *Powers* have borrowed their *Denominations* from *Local Extenſion*.

For a *Line* having but one *Dimenſion*, *viz.* *Length*, drawn into it ſelf, produces a *Square-Plane*.

And that *Square* having two *Dimenſions*, *viz.* *Length* and *Breadth*, drawn into it ſelf, produces a *Cubed-Solid*.

This

This *Cube* has three *Dimenfions, viz. Length, Breadth,* and *Thicknefs*; But the *Nature* and *Property* of *Space* admits of no other *Extenfion.*

Whence it follows, That the *Root* or *Firft Power* being taken as a *Side*, the *Second Power* will be a *Square*, the *Third* a *Cube.*

LEMMA.

Any Quantity (as a Root) *Divided into* Parts *at Pleafure, the Sum of the* Product *of thofe* Parts *drawn into themfelves, and into each other any Number of* Times, *will be equal to that* Quantity *drawn into it felf the like Number of* Times.

Thus, If the Root (r) be made a *Binomial* $a+e$.

Then, $r^2 = a^2 + 2ae + e^2$ } firft } *Power.*
And $r^3 = a^3 + 3a^2e + 3ae^2 + e^3$ } fecond }
&c.

For,

First Power $= a + e =$ Root
$a + e$

$a^2 + ae$
$ae + e^2$

Second Power $= a^2 + 2ae + e^2 =$ Square.
$a + e$

$a^3 + 2a^2e + ae^2$
$a^2e + ae^2 + e^3$

Third Power $= a^3 + 3a^2e + 3ae^2 + e^3 =$ Cube.
&c.

The Method of Proceeding is the fame in the Generating of Higher *Powers*, from any given *Root*, whether *Binomial*, or *Multinomial.*

CO-

COROLLARY.

Now from the Nature of the *Dimension* of each *Species*, their *Place* and *Value* in Numbers are soon Discovered.

As if the Root $\begin{cases} r = a + e \\ 24 = 20 + 4 \end{cases}$

Its Square is $\begin{cases} aa + 2ae + ee = rr. \\ 20 \times 20 + 2 \times 20 \times 4 + 4 \times 4 \\ i.\,e.\ 400 + 160 + 16 \end{cases} = 576$

Its Cube is $\begin{cases} aaa + 3aae + 3aee + eee = rrr \\ 20 \times 20 \times 20 + 3 \times 20 \times 20 \times 4 + 3 \times 20 \times 4 \times 4 + 4 \times 4 \times 4 \\ i.\,e.\ 8000 + 4800 + 960 + 64 = 13824 \end{cases}$

Also for the *Square* of a *Binomial Number*.

Root $75 = 70 + 5 = a + e$

49..	4900	49..	a^2
.70.	.700	35.	$2ae$
..25	..25	.35.	
		25	e^2

Square $5625 = 5625 = 5625 = a^2 + 2ae + e^2$

And for the *Cube*.

Root $75 = 70 + 5 = a + e$

343...	343000	aaa
735..	73500	$3aae$
525.	5250	$3aee$
125	125	eee

Cube $421875 = 421875 = a^3 + 3a^2e + 3ae^2 + e^3$

In

In other Roots *consisting of more Figures, the like Process is to be used.*

1. Taking the Left-hand Figure for (a), and the next to it for (e); then by the *Theorem* for the *Power* set all the Parts in their due Places, and their Sum is the *Power* of those two Figures.

2. These two Figures being taken for (a), the next for (e), and Proceeding as before, you'l have the *Power* of the three Figures; and so on, till you have compleated the *Power* of the whole *Root*. Thus,

For the *Square* or second *Power* of a *Multinomial Number.*

4 6 3 5 7	Root.	
16	= 40000 × 40000	= aa
48	= 40000 × 6000 × 2	= $2ae$
36	= 6000 × 6000	= ee
2116	= 46000 × 46000	= aa
276	= 46000 × 300 × 2	= $2ae$
9	= 300 × 300	= ee
214369	= 46300 × 46300	= aa
4630 . . .	= 46300 × 50 × 2	= $2ae$
25 . .	= 50 × 50	= ee
21483225 . .	= 46350 × 46350	= aa
64890 .	= 46350 × 7 × 2	= $2ae$
49	= 7 × 7	= ee
2148971449	Square.	

For the *Cube*, or *Third Power* of a *Multinomial Number*.

4 6 3	Root
64...	= 400 × 400 × 400 = *aaa*
288..	= 400 × 400 × 60 × 3 = 3 *aae*
432.	= 400 × 60 × 60 × 3 = 3 *aee*
216	= 60 × 60 × 60 = *eee*
97336...	= 460 × 460 × 460 = *aaa*
19044..	= 460 × 460 × 3 × 3 = 3 *aae*
1242.	= 460 × 3 × 3 × 3 = 3 *aee*
27	= 3 × 3 × 3 = *eee*
99252847	Cube

The like Method is to be obferved in the Compofition of other *Powers*, each according to the Nature of the *Theorem* thereto belonging.

SCHOLIUM.

This way of raifing *Powers* from their *Component Parts*, rather leads to the Method of finding the *Root* of a given *Power*, than to find the *Power* of a given *Root*, which is foon done by the *Continual Multiplication* of that *Root*.

CHAP.

CHAP. VII.

Evolution of Quantities.

Definition.

THE Resolution *of* Powers *into their* Roots *is called* Evolution *or the* Analysis *of* Powers.

SCHOLIUM.

And since any *Root* may be considered, as consisting of two Parts, one of which is supposed already known, and the other, tho' unknown, is yet discoverable by means of *Theorems* rais'd by the *Involution* of the *Binomial Root*; in which 'tis very easy to discern how each part of the *Root* is concern'd in the *Power*, and consequently how much is already known, and what remains farther to be enquired for.

Whence, the Method of *Extracting the Root* of any *Power*, may without difficulty be perform'd by observing only the *Constitution* of *Powers*, which plainly directs what manner of Operation each requires.

Now, the first thing to be done, is to distinguish the given Number into several Parts, by *Points* set over such *Places* as the *Index* of the *Power* directs.

Viz. That of *Squares*, *Cubes*, &c. into *Two's*, *Three's*, &c. beginning always from the *Place* of *Units*, and so towards the *Left-hand* in *Integers*, towards the *Right-hand* in *Decimals*; and there will be as many *Places* in the *Roots*, as there are *Points*.

The Method of *Extracting* the *Roots* of the *Second Power*.

Literal EXAMPLES.

I.

$aa + 2ae + ee \; (a + e$

aa

$2a + e) \; 0 + 2ae + ee$

$+ 2ae + ee$

$0 \quad 0$

II.

$a^2 - 6an + 9n^2 + 2ax - 6nx + x^2 \; (a - 3n + x$

aa

$2a - 3n) \; 0 - 6an + 9n^2$

$- 6an + 9n^2$

$2a - 6n + x) \quad 0 + 2ax - 6nx + x^2$

$+ 2ax - 6nx + x^2$

$0 \quad 0 \quad 0$

III.

$a^4 + 4a^3e + 6a^2e^2 + 4ae^3 + e^4 \; (a^2 + 2ae + e^2$

a^4

$2a^2 + 2ae) \; 0 + 4a^3e + 6a^2e^2$

$+ 4a^3e + 4a^2e^2$

$2a^2 + 4ae + e^2) \; 0 \quad + 2a^2e^2 + 4ae^3 + e^4$

$+ 2a^2e^2 + 4ae^3 + e^4$

$0 \quad 0 \quad 0$

Numeral

Numeral *EXAMPLE.*

Theorically thus.

$aa+2ae+ee$			178929 (423
aa =	4×4 =		16....
$2a$ =	2×4 =	8)	.189.. (= e
$2ae$ =	8×2 =		.16..
ee =	2×2 =		...4..
$2a$ =	42×2 =	84)	..2529 (3 = e
$2ae$ =	84×3 =		..252.
ee =	3×3 =		9
			000000

Practically thus.

	178929 (423
82)	.189..
843)	..2529
	0000

The Method of *Extracting* the *Root* of the *Third Power.*

Literal *EXAMPLE.*

$$a^3+3a^2e+3ae^2+e^3\ (a+e$$
$$a^3$$

$$3a^2+3ae+e^2)\ 0+3a^2e+3ae^2+e^3$$
$$+3a^2e+3ae^2+e^3$$
$$0\quad 0\quad 0$$

Numeral

Numeral *EXAMPLE.*

Theorically thus.

$a^3 + 3a^2e + 3ae^2 + e^3 =$			$\dot{7}5\dot{6}86\dot{9}67(423$
a^3	$= 4 \times 4 \times 4$	$=$	64.....
$3a^2$	$= 3 \times 4 \times 4$	$=$ 48)	11686...(2=e
$3a^2e$	$= 3 \times 4 \times 4 \times 2$	$=$	.96.....
$3a^2e$	$= 3 \times 4 \times 2 \times 2$	$=$	..48....
e^3	$= 2 \times 2 \times 2$	$=$	....8...
$3a^2e + 3ae^2 + e^3$		$=$	10088...
$3a^2$	$= 3 \times 42 \times 42$	$=$ 5292)	.1598967 (3=e
$3a^2e$	$= 3 \times 42 \times 42 \times 3$	$=$	.15876..
$3ae^2$	$= 3 \times 42 \times 3 \times 3$	$=$	...1134.
e^3	$= 3 \times 3 \times 3$	$=$	27
$3a^2e + 3ae^2 + e^3$		$=$	.1598967
			.0000000

Practically thus.

75686967 (423

48) 11686

96..
48.
8

5292) 1598967

15876..
1134.
27

0000000

NOTE.

If the Number is not an exact *Square*, *Cube*, &c. Annex to the Remainder, *Two's*, *Three's*, &c. of Cyphers, so any desired Number of *Decimal* Places may be had in that *Root*.

SECT.

SECT. III.

Of the Comparisons *of* Quantities.

CHAP. I.

Of Proportion.

DEFINITION I.

THE Relation *of two* Homogeneous Quantities *one to another, may be considered, either,*

1. By *how much* the one Exceeds the other, which is called their *Difference.*

Thus 5 exceeds 3 by the *Difference* 2.

2. Or *what Part* or *Parts* one is of another, *which is call'd* Ratio.

Thus, the *Ratio* of $\begin{cases} 6 \text{ to } 3 \text{ is } \frac{6}{3} = \frac{2}{1} \text{ or } \textit{Double} \\ 3 \text{ to } 6 \text{ is } \frac{3}{6} = \frac{1}{2} \text{ or } \textit{Subduple.} \end{cases}$

Note, That the Quantity *Compared* is called the *Antecedent*; and that to which it is *Compared* is called the *Consequent.*

DEFINI-

DEFINITION II.

When two $\left\{\begin{array}{l}\text{Differences}\\ \text{Ratio's}\end{array}\right\}$ *are equal, the Terms that Compose them are said to be* $\left\{\begin{array}{l}\text{Arithmetically}\\ \text{Geometrically}\end{array}\right\}$ *Proportional.*

SCHOLIUM I.

Suppose the *Terms* to be a and b, their *Difference* d.

If a be the $\left\{\begin{array}{l}\textit{least}\\ \textit{greatest}\end{array}\right\}$ *Term*, then $\left\{\begin{array}{l}a+d\\ a-d\end{array}\right\} = b$.

Therefore in any *Arithmetic Proportion* when the *Antecedent* is $\left\{\begin{array}{l}\textit{less}\\ \textit{greater}\end{array}\right\}$ than the *Consequent*, the *Terms* may be express'd by a and $\left\{\begin{array}{l}a+d\\ a-d\end{array}\right\}$.

SCHOLIUM II.

Suppose a and b to be the *Terms* of any *Ratio*;

If a be the least *Term*

Put $r = \frac{b}{a}$, then $ar = b$ by *Equal Mult.*

But if b be the least *Term*.

Put $r = \frac{a}{b}$, then $br = a$ by *Equal Mult.*

And $\frac{a}{r} = b$ by *Equal Division.*

Therefore

Therefore, in any *Geometric Proportion*, when the *Antecedent* is $\left\{\begin{matrix}\textit{lefs}\\ \textit{greater}\end{matrix}\right\}$ than the *Confequent*, the *Terms* may be exprefs'd by $\left\{\begin{matrix} a \text{ and } ar\\ a \text{ and } \frac{a}{r}\end{matrix}\right\}$.

DEFINITION III.

THose *Quantities whofe* Excefs *or* Quotients *are the fame, are call'd* Proportionals.

CASE. I.

When of feveral Quantities the $\left\{\begin{matrix}\text{Difference}\\ \text{Quotient}\end{matrix}\right\}$ *of the* 1ft. *and* 2d. *is the fame with that of the* 2d. *and* 3d. *they are faid to be in a* Continued $\left\{\begin{matrix}\text{Arithmetic}\\ \text{Geometric}\end{matrix}\right\}$ Proportion.

Thus, $\left\{\begin{matrix} a, a+d, a+2d, a+3d, a+4d,\\ a, a-d, a-2d, a-3d, a-4d,\end{matrix}\right\}$ &c. is a Series of *Continued Arithmetic Proportionals*, whofe *Common Difference* is *d*.

And $\left\{\begin{matrix} a, & ar, & arr, & arrr, & arrrr, & ar^5\\ a, & \frac{a}{r}, & \frac{a}{rr}, & \frac{a}{rrr}, & \frac{a}{rrrr}, & \frac{a}{r^5}\end{matrix}\right\}$ &c. is a Series of *Continued Geometric Proportionals*, whofe *Common Multiplier* is $\left\{\begin{matrix}\frac{r}{1}\\ \frac{1}{r}\end{matrix}\right\}$ or whofe Ratio is that of $\left\{\begin{matrix}1 \text{ to } r\\ r \text{ to } 1\end{matrix}\right\}$.

Note, That the Sign $\div\!\!\div$ Signifies *Continued Proportion*.

CASE II.

When of several Quantities the $\left\{\begin{array}{l}\text{Difference}\\ \text{Quotient}\end{array}\right\}$ *of the* 1st. *and* 2d. *is the same with that of the* 3d. *and* 4th. (*and not of the* 2d *and* 3d.) *they are said to be in a* Discontinued $\left\{\begin{array}{l}\text{Arithmetic}\\ \text{Geometric}\end{array}\right\}$ Proportion; *such as,*

$$1.\left\{\begin{array}{llll} a, & a+d; & e, & e+d \\ a+d, & a; & e+d, & e \end{array}\right\}$$

$$\text{For } \overline{a+d} - a = \overline{e+d} - e = d.$$

$$2.\left\{\begin{array}{llll} a, & ar; & e, & er; \\ ar, & a; & er, & e \end{array}\right\} \text{for} \left\{\begin{array}{l} \frac{a}{ar} = \frac{e}{er} = \frac{1}{r} \\ \frac{ar}{a} = \frac{er}{e} = \frac{r}{1} \end{array}\right.$$

CHAP. II.

The Chief Properties of Arithmetic Proportion.

THEOREM 1.

In any Number of Continued Arithmetic Proportionals:

1. *If the Number of Terms be even;*

The *Sum* of the *Extreams*, and that of every two equally distant from them, are equal.

2. *If the Number of Terms be odd;*

Then those *Sums* are each equal to the *Double* of the *Middle Term*.

Thus

Thus in, $a,\ a+d,\ a+2d,\ a+3d,\ a+4d.$
'Tis evident that $a+\overline{a+4d}=\overline{a+d}+\overline{a+3d}=2a+4d.$

COROLLARY I.

Therefore, *If three Quantities are in a Continued Arithmetic Proportion*;

The *Sum* of the *Extreams* is equal to the *Double* of the *Middle* Term.

PROBL. I.

To find any Number (n) *of* Arithmetical Mean Proportionals, *between any two given Quantities*, a, *and* e.

Since, $a,\ a+d,\ a+2d,\ a+3d,$ &c. $a+\overline{n+1}\times d=e.$

Theref. $d\times\overline{n+1}=e-a$; and $d=\frac{e-a}{n+1}.$

Or putting $x=$ *First mean* $=$ *Second Term.*

Then $a,\ x,\ 2x-a,\ 3x-2a,\ 4x-3a,$ &c. $x\times\overline{n+1}-na=e$

Therefore $x\times\overline{n+1}=e+na$, and $x=\frac{e+na}{n+1}.$

Now having either the *Difference*, or the *First Mean*, the reft are foon found.

PROB. II.

To find a 3d. 4th. 5th. *or* n*th*. Arithmetic Proportional *to any* two *given* Quantities, a *and* x.

'Tis evident $a+\overline{n-1}\times d$, or $x\times\overline{n-1}-a\times\overline{n-2}$ is the *Proportional* required by the preceding *Problem.*

COROLLARY II.

If in a Rank of *Continued Arithmetic Proportionals*, there be taken any *Series* of Equidistant Terms, That *Series* will be also *Proportional*.

For if, $a, a+d, a+2d, a+3d, a+4d, a+5d, a+6d,$ &c.

Then
$$\begin{cases} a+d, \quad a+3d, \quad a+5d, \quad \text{\&c.} \\ a, \quad a+2d, \quad a+4d, \quad a+6d, \quad \text{\&c.} \\ a, \quad a+3d, \quad a+5d, \quad \text{\&c.} \end{cases}$$
&c.

THEOREM. 2.

If 4 *Quantities be Arithmetically Proportional; The* Sum *of the* Extreams *is equal to the* Sum *of the* Means.

That is, if
$$\begin{cases} a, \; a+d, \; a+2d, \; a+3d, & \textit{Continued.} \\ a, \; a+d; \; e, \; e+d, & \textit{Discontinued.} \end{cases}$$

'Tis plain
$$\begin{cases} a+\overline{a+3d}=\overline{a+d}+\overline{a+2d}=2a+3d \\ a+\overline{e+d}=\overline{a+d}+e. \end{cases}$$

CHAP. III.

The Chief Properties of Geometric Proportion.

THEOREM 1.

In any Number of Continued Geometric *Proportionals:*

1. If the Number of *Terms* be even;

The *Products* of the *Extreams*, and that of every two *Terms*, equally distant from them are *equal*.

2. If the Number of *Terms* be odd;

Then those *Products* which are each equal to the *Square* of the *Middle Term*.

Thus,

Thus, in

$a, ar, ar^2, ar^3, ar^4, ar^5, ar^6,$

'Tis evident that,

$$a \times ar^6 = ar \times ar^5 = ar^2 \times ar^4 = ar^3 \times ar^3 = a^2 r^6.$$

COROLLARY 1.

Hence, *If three Quantities are in a Continual Geometric Proportion*;

The *Product* of the *Extreams* is equal to the *Square* of the *Middle Term.*

Thus in, $a, ar, ar^2.$

'Tis Plain $a \times ar^2 = ar \times ar = a^2 \times r^2.$

PROBLEM I.

To find any Number (n) *of* Geometrical Means *between any* two *given* Quantities, a *and* e.

Since, $a, ar, ar^2, ar^3,$ &c. $ar^{n+1} = e$

Therefore, $r^{n+1} = \frac{e}{a}$, And $r = \overline{\frac{e}{a}}\Big|^{\frac{1}{n+1}} =$ *Ratio.*

Or, putting $x =$ *First Mean* $=$ *Second Term.*

Then, $a, x, \frac{x^2}{a}, \frac{x^3}{a^2}, \frac{x^4}{a^3},$ &c. $\frac{x^{n+1}}{a^n} = e$

Therefore, $x^{n+1} = a^n e$, And $x = \overline{a^n e}\big|^{\frac{1}{n+1}}$.

Now, having found either the *Ratio* or *First Mean*, the rest are soon got.

PROB.

PROBLEM II.

Given, *any two* Quantities, a *and* x.
Required, *a* 3d, 4th, 5th, *or* *n*th, Geometric Proportion.

'Tis evident from the Preceding *Problem*, that $a \times r^{n-1}$, or $\frac{x^{n-1}}{a^{n-2}}$ is the *Proportional* required.

COROLLARY 2.

If out of a Rank of *Continued Geometric Proportionals*, there be taken any *Series* of Equidistant *Terms*, that *Series* shall be also *Proportional*.

Thus,

If, $a,\ ar,\ ar^2,\ ar^3,\ ar^4,\ ar^5,\ ar^6,\ ar^7,\ ar^8$

$$\text{Then}\left\{\begin{array}{l} a,\ ar^2,\ ar^4,\ ar^6,\ ar^8 \\ ar,\ ar^3,\ ar^5,\ ar^7, \\ a,\ ar^3,\ ar^6, \\ a,\ ar^4,\ ar^8 \end{array}\right\} \text{\&c.} \div$$

&c.

THEOREM 2.

If 4 Quantities be Geometrically Proportional;
The *Product* of the *Extreams*, is equal to that of the *Means*.

That is, if $\left\{\begin{array}{l} a,\ ar,\ ar^2,\ ar^3, \\ a,\ ar,\ e,\ er \end{array}\right.$ Continued / Discontinued

'Tis plain that $\left\{\begin{array}{l} a \times ar^3 = ar \times ar^2\ (= a^2 r^3) \\ a \times er = ar \times e\ \ (= aer = are). \end{array}\right.$

Or,

Or, if $3 : 5 :: 2 \times 3 : 2 \times 5$
i. e. $3 : 5 :: 6 : 10$

Then $3 \times 10 = 5 \times 6$
i. e. $3 \times 2 \times 5 = 5 \times 2 \times 3$ $\Big\} = 30.$

CONVERSE.

If 2 Products arising from the Multiplication of two Quantities be equal;
Those 4 Quantities will be Proportional.

CASE I.

In any two Products, *if the* Factors *of the* one *be made the* 1st. Antecedent, *and the* 2d. Consequent; *those of the* other, *the* 1st. Consequent, *and* 2d. Antecedent, *then the* Terms *are said to be* Directly Proportional.

That is, if $Ab = aB$.

$$\text{Then}\begin{cases} A : B :: a : b, \text{ or } A : a :: B : b. \\ B : A :: b : a, \text{ or } B : b :: A : a. \\ a : b :: A : B, \text{ or } a : A :: b : B. \\ b : a :: B : A, \text{ or } b : B :: a : A. \end{cases}$$

SCHOLIUM.

If in 4 Quantities *directly Proportional* ($A : a :: B : b$) any *Three* be given, the *Fourth* is readily found.

For $Ab = Ba$, *i. e.* *1st.* $\times$ *4th.* $=$ *2d.* $\times$ *3d.* by this *Theorem.*

Therefore

Theref. by equal *Division.*

$$\begin{cases} 1st. = \frac{2^d \times 3^d}{4^{th}} = 2d. \times \frac{3^d}{4^{th}} = 3d. \times \frac{2^d}{4^{th}} \\ 2d. = \frac{1^{st} \times 4^{th}}{3^d} = 1st. \times \frac{4^{th}}{3^d} = 4th. \times \frac{1^{st}}{4^{th}} \\ 3d. = \frac{1^{st} \times 4^{th}}{2^d} = 1st. \times \frac{4^{th}}{2^d} = 4th. \times \frac{1^{st}}{2^d} \\ 4th. = \frac{2^d \times 3^d}{1^{st}} = 2d. \times \frac{3^d}{1^{st}} = 3d. \times \frac{2^d}{1^{st}} \end{cases}$$

NOTE.

That $\frac{Ba}{A}$ is a 4th *Proportional* to A, B, and *a*, may be proved also thus;

For since $A : Ba :: 1 : \frac{Ba}{A}$ by *Def.* of *Division.*

And $A \times \frac{Ba}{A} (= Ba \times 1) = Ba$ by this *Theor.*

Therefore, $A : B :: a : \frac{Ba}{A}$, by its *Converse*, *q. e. d.*

CASE II.

But if the Factors *of the* one *be made the* 1st. Antecedent, *and* 1st. Consequent; *those of the* other, *the* 2d. Antecedent, *and* 2d. Consequent, *then the* Terms *are said to be* Reciprocally Proportional.

That is, if $Ab = aB$.

Then
$$\begin{cases} A : b :: B : a, \text{ or } A : b :: a : B. \\ B : a :: A : b, \text{ or } B : a :: b : A. \\ a : B :: b : A, \text{ or } a : B :: A : b. \\ b : A :: a : B, \text{ or } b : A :: B : a. \end{cases}$$

SCHO-

SCHOLIUM.

If in 4 Quantities *Reciprocally Proportional* (A : *b* : : *a* : B) any *Three* be given, the 4th is eafily found.

For A*b* = *a*B, *i. e.* 1ft. × 2d. = 3d. × 4th. by the *Def.* of *Recipr. Proportion.*

Theref. by equal *Division*,

$$\begin{cases} 1ft. = \frac{3^d \times 4^{th}}{2^d} = 3d. \times \frac{4^{th}}{2^d} = 4th. \times \frac{3^d}{2^d} \\ 2d. = \frac{3^d \times 4^{th}}{1^{ft}} = 3d. \times \frac{4^{th}}{1^{ft}} = 4th. \times \frac{3^d}{1^{ft}} \\ 3d. = \frac{1^{ft} \times 2^d}{4^{th}} = 1ft. \times \frac{2^d}{4^{th}} = 2d. \times \frac{1^{ft}}{4^{th}} \\ 4th. = \frac{1^{ft} \times 2^d}{3^d} = 1ft. \times \frac{2^d}{3^d} = 2d. \times \frac{1^{ft}}{3^d} \end{cases}$$

COROLLARY 2.

In Ranks of *Similar Proportions*, the *Sums* or *Difference* of the Correfponding *Terms* fhall be *Proportional.*

That is, if $\begin{cases} a : ar :: e : er :: \\ x : xr :: z : zr \end{cases}$ &c.

Then $a \pm x : ar \pm xr :: e \pm z : er \pm zr.$

COROLLARY 3.

In two Ranks of *Proportionals*, the *Products* or *Quotients* of the Correfponding *Terms* will be *Proportional.*

That is, if $\begin{cases} a : ar :: e : er \\ x : xs :: z : zs \end{cases}$

 Then

$$\text{Then}\begin{cases} ax : arxs :: ez : erzs \\ \dfrac{a}{x} : \dfrac{ar}{xs} :: \dfrac{e}{z} : \dfrac{er}{zs} \end{cases}$$

COROLLARY 4.

The *Powers* of *Proportionals*, are allo *Proportional.*

That is,

Continued — If $a,\ ar,\ ar^2,\ ar^3$ &c.; *Discontinued* — $a,\ ar;\ e,\ er,$ *Roots.*

$$\text{Then}\begin{cases} a^2, a^2r^2, a^2r^4, a^2r^6 \\ a^3, a^3r^3, a^3r^6, a^3r^9 \\ \text{\&c.} \end{cases} \ \&\ \begin{cases} a^2, a^2r^2;\ e^2, e^2r^2 & \textit{Squares.} \\ a^3, a^3r^3;\ e^3, e^3r^3 & \textit{Cubes.} \\ \text{\&c.} \end{cases}$$

COROLLARY 5.

Products having the same Common *Factor* are as the other *Factors.*

$$\text{That is,}\begin{cases} an : en :: a : e. \\ anm : enm :: a : e. \\ \text{\&c.} \end{cases}$$

For $an \times e = en \times a$; and $anm \times e = enm \times a$, &c.

SCHOLIUM 1.

Hence, The *Product* of any two *Quantities* a and e, is a *mean Proportional* between the *Squares* of those *Quantities.*

Thus $aa : ae :: ae : ee$.
For *P. Means* = *P. Extreams.*

SCHO-

SCHOLIUM 2.

Therefore, between a^n and e^n, there are $n-1$ *Mean Proportionals* Composed of two Ranks of *Powers*,

The $\left\{\begin{matrix}\text{one } \textit{In-}\\ \text{other } \textit{De-}\end{matrix}\right\}$ *creasing*, from $\left\{\begin{matrix} a^{n-1} \text{ to } a^{n-n} \\ e^{n-n} \text{ to } e^{n-1}\end{matrix}\right\}$.

SCHOLIUM 3.

If *Unit* be put for the 1st. *Term*, the *Root* for the 2d. and its Consequent *Powers* for the 3d. 4th, &c. *Terms* (in order), they will be a *Series of Geometric Proportionals*, whose *Common Ratio* is the *Root* or *Second Term*; And their *Exponents* will be a Series of *Arithmetic Proportionals*, (expressing the *Dimension* of each *Term*, or its Distance from *Unity*) either *Affirmative* or *Negative*, according as the *Root* is *More* or *Less* than *Unit*.

Thus $\left\{\begin{matrix} +4 & +3 & +2 & +1 & 0 & -1 & -2 & -3 & -4 \\ a, & a, & a, & a, & 1, & a, & a, & a, & a, \end{matrix}\right.$ &c. $\left\{\begin{matrix}\textit{Expon.}\\ \textit{Powers}\end{matrix}\right.$

But $\frac{a^4}{1}, \frac{a^3}{1}, \frac{a^2}{1}, \frac{a}{1}, 1, \frac{1}{a}, \frac{1}{a^2}, \frac{1}{a^3}, \frac{1}{a^4}$, ∴

Therefore, $a^{-n} = \frac{1}{a^n}$.

Hence it is, that a *Quantity* drawn into it self any *Number* of *Times*, that *Number* more *one* is the *Exponent* thereof.

Therefore, to $\left\{\begin{matrix}\textit{Double}\\ \textit{Triple}\\ \textit{\&c.}\end{matrix}\right\}$ the *Index* of any *Power*, is to $\left\{\begin{matrix}\textit{Square}\\ \textit{Cube}\\ \textit{\&c.}\end{matrix}\right\}$ that *Power*.

Conseq. to take $\left\{\begin{matrix}\frac{1}{2}\\ \frac{1}{3}\\ \textit{\&c.}\end{matrix}\right\}$ of the *Index* of any *Power*, is to *Extract* the $\left\{\begin{matrix}\textit{Square}\\ \textit{Cube}\\ \textit{\&c.}\end{matrix}\right\}$ *Root* of that *Power*.

Therefore, $\sqrt[r]{a^n}$ is more Naturally express'd by $a^{\frac{n}{r}}$.

 Or,

Or, Because the *Root* is the 1st. *Geometrical Mean* between 1 and the *Power*;

But $\sqrt[r]{a^n}$ is the 1st. *Geometrical Mean* between the Powers 1 and a^n. by *Probl.* 1. *Ch.* 3.

And $\frac{n}{r}$ is the 1st. *Arithmetical Mean* between their *Exponents* o and n. by *Probl.* 1. *Ch.* 2.

Therefore $\frac{n}{r}$ is the *Exponent* of the *Power*.

$$\text{That is, } \sqrt[r]{a^n} \; (= \overline{a^n}|^{\frac{1}{r}}) = a^{\frac{n}{r}}.$$

$$\text{Also, } \frac{1}{\sqrt[r]{a^n}} \; (= \frac{1}{\overline{a^n}|^{\frac{1}{r}}}) = a^{-\frac{n}{r}}$$

$$\text{And } \frac{a^n}{\sqrt[r]{a^m}} = \frac{a^n}{\overline{a^m}|^{\frac{1}{r}}} = a^n \times \frac{1}{\overline{a^m}|^{\frac{1}{r}}} = a^{n-\frac{m}{r}} = a^{\frac{nr-m}{r}}$$

For the same *Reason*.

COROLLARY 6.

If $Am : an :: B : b$, then $Amb = anB$

Theref. $A : a :: Bn : bm$, for $Abm = aBn$.

COROLLARY 7.

If $a : ar :: e : er$

Then 1. $na : ar :: ne : er$

2. $na : ar :: e : \frac{er}{n}$

3. $a : \frac{ar}{n} :: ne : er$.

COROLLARY 8.

If $A : a :: B : b$

i.e. $A : Ar :: B : Br$.

Then

Then 1. $AA : aa :: AB : ab$.
2. $Aa : Ab :: Ab : Bb$.
3. $AA : aB :: AB : Bb$.
4. $AB : Aa :: BB : Ab$.

COROLLARY 9.

If $\begin{cases} a : ar :: e : er. \\ x : xr :: e : er. \end{cases}$

Then $a \pm x : ar \pm xr :: e : er$.

COROLLARY 10.

If $\begin{cases} A : B :: a : b \\ C : B :: a : d \end{cases}$ then $A : C :: d : b$.

For Ab ($= Ba$) $= Cd$, by this *Theor.*
Therefore $A : C :: d : b$. by its *Converse*.

COROLLARY 11.

In $\begin{cases} A, B, C \\ a, b, c \end{cases}$ if $\begin{cases} A : B :: b : c \\ B : C :: a : b \end{cases}$

Then $A : C :: a : c$.
For Ac ($= Bb$) $= Ca$, by this *Theorem*.
Therefore $A : C :: a : c$ by its *Converse*.

COROLLARY 12.

If $a : ar :: e : er$.

Then $\begin{cases} a : ar \\ e : er \end{cases} :: a \pm e : ar \pm er$.

COROLLARY 13.

In any Number of *Geometric Proportionals*;

As one Antecedent is to its Consequent;
So is the Sum of the Antecedents, to the Sum of the Consequents.

i. e.

i. e. If $\begin{cases} a,\ ar,\ ar^2,\ ar^3,\ \&c. & \textit{Continued,} \\ a,\ ar;\ e,\ er,\ \&c. & \textit{Discontinued,} \end{cases}$

Then,

$$a : ar :: a + ar + ar^2 : \begin{cases} \overline{a + ar + ar^2} \times r \\ ar + ar^2 + ar^3 \end{cases}$$

$$a : ar :: a + e : \begin{cases} \overline{a + e} \times r \\ ar + er \end{cases}$$

COROLLARY 14.

If $A : a :: B : b$. Then

1. *Alternately*, $A : B :: a : b$.
2. *Inversly*, $a : A :: b : B$.
3. *Compound*. $A + a : a :: B + b : b$.
4. *Conversly*, $A : A \pm a :: B : B \pm b$.
5. *Dividedly*, $A - a : a :: B - b : b$.
6. *Mixtly*, $A + a : A - a :: B + b : B - b$.

All evidently *Proportional*, since the *Products* of their *Extreams* and *Means* are the *same* with those of the given *Proportion*.

SCHOLIUM.

Proportions Compounded, *Converted*, *Divided* and *Mixt*, are also *Proportionals* when *Alterned* and *Inverted*.

COROLLARY 15.

If $\begin{cases} a : ar :: e : er \\ e : er :: u : ur \end{cases}$

Then $a : ar :: u : ur$.

SCHOLIUM.

In $\left\{\begin{matrix} A, B, C, \\ a, b, c, \end{matrix}\right\}$ if $\left\{\begin{matrix} A : B :: a : b \\ B : C :: b : c \end{matrix}\right.$

Then $A : C :: a : c$.

$$\left.\begin{matrix} \text{For } A : a :: B : b. \\ \text{And } B : b :: C : c. \end{matrix}\right\} \text{by } Altern.$$

Therefore $A : a :: C : c$. by this *Cor.*

And $A : C :: a : c$. by *Altern.*

COROLLARY 16.

If $\left\{\begin{matrix} a : ar :: e : er \\ a : ar :: u : ur \end{matrix}\right\}$

Then $e : er :: u : ur$.

DEFINITION IV.

THE *Product of the* like *Terms of any* Ratio *is called a* Compound Ratio *or a* Ratio *compounded of those* Ratio's.

Thus, the *Ratio* $\frac{ae}{mn}$ is *Compounded* of $\frac{a}{m}$ and $\frac{e}{n}$, or of $\frac{a}{n}$ and $\frac{e}{m}$.

SCHOLIUM.

Whence, *Compound Ratio's* are produced by the *Multiplication* of the *Ratio's* compounding them;

That is, $\frac{ae}{mn} = \frac{a}{m} \times \frac{e}{n}$.

For

For $mnae = anme$ by *Schol.* 4. *Ch.* 5. §. 2.
And $mn : an :: me : ae$ by *Conv. Theor.* 2.

Theref. $\frac{mn}{mn} : \frac{an}{mn} :: \frac{me}{mn} : \frac{ae}{mn}$ by *equal Divis.*

i. e. $1 : \frac{a}{m} :: \frac{e}{n} : \frac{ae}{mn}$

Confeq. $\frac{ae}{mn} = \frac{a}{m} \times \frac{e}{n}$. by *Theor.* 2.

Definition V.

Similar Products *are those whose Corresponding* Factors *are Proportional.*

Theorem 3.

All Similar Products *are as the Terms of the* Ratio *of the Corresponding* Factors, *rais'd into the* Product's *Dimension.*

Suppose A B C, &c. and $a\ b\ c$, &c. the *Products*, whose Number of *Dimension* is n.

The *Similar Factors* being as r to s.

That is $\begin{cases} A : a :: r : s \\ B : b :: r : s \\ C : c :: r : s \\ \text{\&c.} \end{cases}$

Theref. ABC, &c. : abc, &c. :: $r^n : s^n$. by *Cor.* 3. *Theor.* 2.

Corollary I.

The *Ratio* of any two *Products* is Composed of the *Ratio's* of the *Factors*.

COROLLARY 2.

And all *Similar Powers* are in a *Ratio* Compounded of their *Roots* rais'd into the *Index* of these *Powers*.

COROLLARY 3.

Also *Ratio's* Compounded of equal ones are equal to one another.

This, if $\frac{a}{m} = \frac{e}{n}$, and $\frac{x}{y} = \frac{u}{z}$, then $\frac{ax}{my} = \frac{eu}{nz}$.

THEOREM 4.

In any given Quantities, (a, b, c, d, e,) *the* Ratio *of the* Extreams *is Compounded of all the Intermediate ones.*

That is, $\frac{a}{e} = \frac{a}{b} \times \frac{b}{c} \times \frac{c}{d} \times \frac{d}{e} = \frac{abcd}{bcde}$.

COROLLARY.

Therefore, If $\left\{\begin{array}{l} A : B :: a : b \\ B : C :: b : c \\ C : D :: c : d \end{array}\right\}$ &c. Then $A : D :: a : d$.

For $\frac{A}{B} = \frac{a}{b}$, $\frac{B}{C} = \frac{b}{c}$, $\frac{C}{D} = \frac{c}{d}$, by *Def.* of *Proport.*

And $\frac{A}{B} \times \frac{B}{C} \times \frac{C}{D} = \frac{a}{b} \times \frac{b}{c} \times \frac{c}{d}$.

That is, $\frac{A}{D} = \frac{a}{d}$, by this *Theor.*

Conseq. $A : D :: a : d$. by *Def.* of *Proport.*

Definition VI.

ALL Ratio's *Compounded of* 2, 3, *&c. Equal ones, are called the* Duplicate, Triplicate, *&c. of any one of those* Ratio's.

Thus, if $\frac{a}{m} = \frac{e}{n}$, then $\frac{ae}{mn} = \frac{aa}{mm} = \frac{ee}{nn}$ is *Duplicate* of $\frac{a}{m}$, or $\frac{e}{n}$.

Corollary.

Therefore, In any Number of *Continued Geometric Proportionals,*

$1,\ a,\ a^2,\ a^3,\ a^4,\ a^5,\ a^6,$ &c.

The *Ratio* of the 1*st.* to the $\left\{\begin{array}{l}\text{3d.}\\ \text{4th.}\\ \text{5th.}\\ \text{\&c.}\end{array}\right\}$ is $\left\{\begin{array}{l}\textit{Duplicate}\\ \textit{Triplicate}\\ \textit{Quadrupl.}\\ \text{\&c.}\end{array}\right\}$ of that of the 1st. to the 2d.

And Consequently,

The *Ratio* of the $\left\{\begin{array}{l}\text{3d.}\\ \text{4th.}\\ \text{5th.}\end{array}\right\}$ to the 1st. is $\left\{\begin{array}{l}\textit{Duplicate}\\ \textit{Triplicate}\\ \textit{Quadrupl.}\end{array}\right\}$ of that of the 2d. to the 1st.

Scholium.

In any *Series* of *Geometric Proportionals;*

$a,\ ar,\ ar^2,\ ar^3,\ ar^4,\ ar^5,\ ar^6,$ &c.

As $a^n : \overline{ar}|^n :: a : a \times r^n$, that is,

As

As any *Power* of the 1ſt. is to the like of the 2d.
So is the 1ſt. to the $n + 1$ *Term*:
Or, So is the 1ſt. 2d. 3d. &c. to the *n*th. *Term* from it.

LEMMA 1.

To Multiply *a* Ratio *by any Quantity*, is to Multiply *the* Antecedent *of that* Ratio *by that Quantity*.

That is, $m \times \frac{A}{B} = \frac{mA}{B}$.

For $B : A :: m : \frac{mA}{B}$ by *Sch.* to *Conv. Theor.* 2.

And $B : A :: 1 : \frac{A}{B}$ by *Def.* of *Diviſ.*

Theref. $1 : \frac{A}{B} :: m : \frac{mA}{B}$ by *Cor.* 16. *Theor.* 2.

But $1 : \frac{A}{B} :: m : \frac{A}{B} \times m$, by *Def.* of *Mult.*

Conſeq. $\frac{A}{B} \times m = \frac{mA}{B}$.

THEOREM. 5.

Ratio's *having the* ſame Conſequents *are* Directly *as their* Antecedents.

That is, $\frac{A}{B} : \frac{a}{B} :: A : a$

For $a \times \frac{A}{B} = \frac{aA}{B}$, and $A \times \frac{a}{B} = \frac{Aa}{B}$ } by *Lem.* 1.
But $\frac{aA}{B} = \frac{Aa}{B}$, or $a \times \frac{A}{B} = A \times \frac{a}{B}$ }

Theref. $\frac{A}{B} : \frac{a}{B} :: A : a$. by *Conv. Theor.* 2.

LEMMA 2.

If a Quantity Multiplies *another, and the* Product *be* Divided *by the same, the* Quotient *will be the Quantity* Multiplied.

That is, $a = \frac{an}{n} = \frac{anr}{nr}$, &c.

For $1 : a :: n : an$. by *Def.* of *Mult.*

And $n : an :: 1 : \frac{an}{n}$ by *Def.* of *Divis.*

Theref. $1 : a :: 1 : \frac{an}{n}$ by *Cor.* 15. *Theor.* 2.

Conseq. $a = \frac{an}{n}$, &c. by *Def.* of *Proport.*

COROLLARY.

Therefore If the *Terms* of a *Ratio* be *Multiplied* or *Divided* by the same Quantity, the *Ratio* will be the *same.*

That is, $\frac{x}{n} = \frac{xz}{nz}$, &c.

For putting $an = x$, then $\frac{an}{n} = \frac{x}{n}$, and $\frac{anz}{nz} = \frac{xz}{nz}$

But $\frac{an}{n}$ $(= a) = \frac{anz}{nz}$ by this *Lemma.*

Therefore $\frac{x}{n} = \frac{xz}{nz}$, &c. by *Substitution.*

THE-

THEOREM 6.

Ratio's *having the* same Antecedents *are* Reciprocally *as their* Consequents.

That is $\frac{A}{B} : \frac{A}{C} :: C : B$.

For $\frac{A}{B} = \frac{AC}{BC}$, and $\frac{A}{C} = \frac{BA}{BC}$ by preced. *Cor.*

But $\frac{AC}{BC} : \frac{BA}{BC} :: AC : BA$, by *Theor.* 5.

That is, $\frac{A}{B} : \frac{A}{C} :: C : B$.

THEOREM 7.

If $A : B :: a : b$

Then $\left\{ \begin{array}{l} nA : B :: na : b \\ A : nB :: a : nb \end{array} \right\}$ and $\left\{ \begin{array}{l} \frac{A}{n} : B :: \frac{a}{n} : b \\ A : \frac{B}{n} :: a : \frac{b}{n} \end{array} \right.$.

1. For $A : a :: B : b$, by *Altern.*
And $nA : na :: A : a$, by *Cor.* to. *Lem.* 2.
Theref. $nA : B :: na : b$, by *Altern.*

2. Also $\frac{A}{n} : \frac{a}{n} :: A : a$ by *Theor.* 5.

Theref. $\frac{A}{n} : B :: \frac{a}{n} : b$, by *Altern.*

THE-

THEOREM 8.

Quantities Proportional *to their* Differences *are Continually* Proportional.

For if $a : a-b :: b : b-c$

Then $a - \overline{a-b} : a-b :: b - \overline{b-c} : b-c$, by *Divid.*

That is $b : a-b :: c : b-c$

And $b : c :: a-b : b-c$ } by *Altern.* of { *Last* / *First* } *Step.*
But $a : b :: a-b : b-c$

Theref. $a : b :: b : c$. by *Cor.* 16. *Theor.* 2.

THEOREM 9.

If Quantities are Continually Proportional, *their* Sums *or* Differences *are also in* Continual Proportion.

For if $a : b :: b : c :: c : d :: d : e$, &c.

Then $a : a \pm b :: b : b \pm c :: c : c \pm d :: d : d \pm e$.

And $\left\{ \begin{array}{l} a : b :: a \pm b : b \pm c \\ b : c :: b \pm c : c \pm d \end{array} \right\}$ by *Altern.*
&c.

But $a : b :: b : c :: c : d$, &c. by *Sup.*

Theref. $a \pm b : b \pm c :: b \pm c : c \pm d$, &c.

THEOREM 10.

If $A : B :: a : b$; then $A : \overline{AB}|^{\frac{1}{2}} :: a : \overline{ab}|^{\frac{1}{2}}$.

For $AA : AB :: a : b$ } by *Cor. Lem.* 2.
And $AA : AB :: aa : ab$

Theref. $A : \overline{AB}|^{\frac{1}{2}} :: a : \overline{ab}|^{\frac{1}{2}}$, by *Evolution.*

CHAP.

CHAP. IV.

Of Harmonic *and* Contra-Harmonic Proportions.

DEFINITION I.

When 3 Terms are so disposed, that the

$$1st. \backsim 2d. : 2d. \backsim 3d. :: 1st. : 3d.$$

they are said to be Harmonically Proportional.

Thus, 10, 15, 30, are *Harmonic. Proportional.*

For $10 \backsim 15 : 15 \backsim 30 :: 10 : 30$.

And $b^2 - bn$, $b^2 - n^2$, $b^2 + bn$, make an *Harm. Prop.*

For $bn - n^2 : bn + n^2 :: b^2 - bn : b^2 + bn$.

Also, 12, 6, 4, are *Harmon. Proport.*

For $12 - 6 : 6 - 4 :: 12 : 4$.

So $b^2 + 3bn + 2n^2$, $b^2 + 2bn$, $b^2 + bn$, are *Har. Prop.*

For $bn + 2n^2 : bn :: b^2 + 3bn + 2n^2 : b^2 + bn$.

COROLLARY.

Whence, If the 2 first Terms of an *Harmonic Proportion* be *given*, the 3d. is readily found.

For if a, b, c, be *Harmonically Proportional.*

Then, $a - b : b - c :: a : c$, and $ac - bc = ab - ac$.

Theref. $ab = \overline{2a - b} \times c$, and $bc = \overline{2c - b} \times a$.

Conseq. $c = \frac{ab}{2a - b}$, and $a = \frac{bc}{2c - b}$.

DEFI-

DEFINITION II.

WHen 4 *Terms are so disposed, that the*

$$1st. \backsim 2d. : 3d. \backsim 4th. :: 1st. : 4th.$$

they are also Harmonically Proportional.

As 10, 16, 24, 60; For $10 \backsim 16 : 24 \backsim 60 :: 10 : 60$.

COROLLARY.

If the 3 first Terms of such an *Harmonic Proportion* be given, the 4th is easily found.

For if a, b, c, d, be *Harmon. Proportional.*

Then $a - b : c - d :: a : d$; and $ad - bd = ac - ad$

Theref. $d = \frac{ac}{2a - b}$, and $a = \frac{bd}{2d - c}$.

DEFINITION III.

IF *the Terms of an* Harmonic Proporti-on *be continued, then 'tis called an* Harmonic Progression.

Thus, Supposing { b to be the 2*d.* Term; d the Difference of the 1*st.* and 2*d.* }

And that the 1*st.* exceeds the 2*d.*

The *Progression* will be

$$b + d,\ b,\ \frac{b^2 + bd}{b + 2d},\ \frac{b^2 + bd}{b + 3d},\ \frac{b^2 + bd}{b + 4d},\ \frac{b^2 + bd}{b + 5d},\ \&c.$$

COROL

COROLLARY 3.

Whence, If out of a Rank of *Harmonic Proportionals* there be taken any *Series* of equidistant Terms, that *Series* will be *Harmonically Proportional.*

SCHOLIUM.

'Tis observed also, that *Harmonic Proportion* has several other *Properties* common with those of *Arithmetic* and *Geometric Proportions.*

DEFINITION IV.

WHen three Terms *are so disposed, that the* Diff. *of the* 1st. *and* 2d. : Diff. *of the* 2d. *and* 3d. : : 3d. : 1st. *they are said to be in a* Contra-Harmonic Proportion.

Thus, 6, 5, 3, and 12, 10, 4, are *Contra-Harmonics.* For 6 — 5 : 5 — 3 : : 3 : 6; and 12 — 10 : 10 — 4 : : 4 : 12

Or supposing b greater than n.

If the 2*d*. *Term* be greater than the 1*st*:

Then $bn + n^2$, $b^2 + n^2$, $b^2 + bn$, are *Contra-Harmon.*

For $bn - b^2 : n^2 - bn :: b^2 + bn : bn + n^2$.

But if the 1*st*. *Term* exceeds the 2*d*:

Then $b^2 + bn$, $b^2 + n^2$, $bn + n^2$, are *Contra-Harmon.*

For $bn - n^2 : b^2 - bn :: bn + n^2 : b^2 + bn$.

CHAP. V.

Of Fractions or Parts of Integers.

DEFINITION I.

ANY Whole Thing *or an* Unit, *may be considered as* Divided *into any Number of* Equal Parts, *which have their* Name *from the* Number *contain'd in that* Unit.

As if an *Unit* be conceiv'd to be divided into *four* equal *Parts*, those *Parts* are call'd *Fourths*, and are thus written, $\frac{}{4}$.

And any *Number* of those *Parts*, suppose *three* are thus express'd, *Three Fourths*, or *Three divided by Four*, and thus written $\frac{3}{4}$, and called a *FRACTION*.

SCHOLIUM 1.

Hence, the *Number* {*above* / *below*} the Line is call'd the {*Numerator* / *Denominator*} expressing {*some of* / *all*} those *Parts* into which the *Unit* is *Divided*.

SCHOLIUM 2.

If the *Numerator* be {*greater* than / *equal* to / *less* than} the *Denominator*, the *Fraction* is {*greater* than / *equal* to / *less* than} *Unit*, and is called {an *Improper* / a *Proper*} *Fraction*.

COROL-

COROLLARY 1.

Whence it follows, That a *Fraction* is but a *Quotient*, signifying a *Part* or *Parts* of an *Unit*, expreſs'd by a *Numerator* as a *Dividend* and a *Denominator* as a *Diviſor*.

Thus, *one Third* part of any *Thing* is the *Quotient* of that *Thing Divided* by *Three*, which by *Common Diviſion* is expreſſed thus, $\frac{1}{3}$.

Alſo $\frac{3}{4}$ ſignifies *Three Fourth* parts of an *Unit*, or *One Fourth* part of *Three Units*.

For $\left\{\begin{matrix}1\\3\end{matrix}\right\}$ *fourth* of $\left\{\begin{matrix}3 \text{ Units}\\1 \text{ Unit}\end{matrix}\right\}$ is *Thrice one fourth* of *one Unit*.

COROLLARY 2.

As the *Numerator* is to the *Denominator*;
So is the *Fraction* to *Unit*.

For the *Dividend* is to the *Diviſor*,
As the *Quotient* is to *Unit*. by *Def.* of *Diviſion*.

COROLLARY 3.

Fractions having the ſame *Denominators* are one to another as their *Numerators*.

Thus, $\frac{2}{5} : \frac{3}{5} :: 2 : 3$.

For $2 : \frac{2}{5} :: 5 : 1$
And $3 : \frac{3}{5} :: 5 : 1$ } by preced. *Cor.*

Theref. $\frac{2}{5} : \frac{3}{5} :: 2 : 3$, by *Cor.* 16. *Th.* 2. *Ch.* 3

COROLLARY 4.

As any Fraction is to Unit,
So is Unit to the Reverse Fraction.

Thus, $\frac{3}{4} : 1 :: 1 : \frac{4}{3}$.
For $\frac{3}{4} : 1 :: 3 : 4$ } by *Cor.* 2.
And $1 : \frac{4}{3} :: 3 : 4$ }
Theref. $\frac{3}{4} : 1 :: 1 : \frac{4}{3}$. by *Cor.* 16. *Th.* 2. *Ch.* 3.

LEMMA 1.

To Multiply the *Numerator* is to Multiply the *Fraction*; thus, $\frac{2 \times 3}{5} = \frac{2}{5} \times 3$.

For $5 : 2 :: 3 : \frac{2 \times 3}{5}$ by *Case* 1. of *Conv.* to *Th.* 2.
And $5 : 2 :: 1 : \frac{2}{5}$, by *Def.* of *Divis.*
Theref. $1 : \frac{2}{5} :: 3 : \frac{2 \times 3}{5}$ by *Cor.* 16. *Th.* 2. *Ch.* 3.
But $1 : \frac{2}{5} :: 3 : \frac{2}{5} \times 3$, by *Def.* of *Mult.*
Theref. $\frac{2 \times 3}{5} = \frac{2}{5} \times 3$.

LEMMA 2.

The *Terms* of a *Fraction* being *Multiplied* or *Divided* by the same *Quantity* alter not its value.

Thus $\frac{2 \times 3}{2 \times 4} = \frac{6}{8}$

For $2 \times 3 \ (= 6) : 2 \times 4 \ (= 8) :: 3 : 4$, by *Cor.* 5. *Th.* 2.
Theref. $\frac{3}{4} = \frac{6}{8}$, by *Def.* of *Proportion.*

Also

Also, $\frac{5)\ 10\ (2}{5)\ 15\ (3}$

For, $\frac{10}{5} : \frac{15}{5} :: 10 : 15$, by *Cor.* 3. *Def.* 1.

But, $\frac{10}{5} : \frac{15}{5} :: 2 : 3$.

Theref. $\frac{10}{15} = \frac{2}{3}$, by *Def.* of *Proportion.*

Definition II.

'TIS *not only an* Unit *that may be Divided into any Number of* Equal Parts, *but also any of those* Parts *may be divided infinitly into others call'd* Compound Fractions, *and those again subdivided infinitly.*

These things being throughly understood, all Operations relating to *Fractions* admit of very few or no Difficulties.

Definition III.

THE *Changing of Quantities out of one* Form *or* Denomination *into another, (either for the more ease in* Working, *or* Estimating *of their* Value) *is called* Reduction.

SCHO-

SCHOLIUM.

Since it often happens in *Reducing*, *Adding*, *Subtracting*, &c. *Fractions*, that they ſwell into too great Numbers, which are not ſo managable as ſmaller ones; therefore we ſhall in the next place ſhew the way of *Reducing* them into their *Leaſt Terms*, either before or after ſuch Operations, as there is occaſion; which is done by the help of the following

PROBLEM.

Two Numbers being given, to find their Greateſt Common Meaſure, (i. e.) *the Greateſt Number that can* Divide *both without* Remainder.

RULE.

Divide the Greateſt by the Leaſt, and that Diviſor by the Remainder continually, till nothing remain, and the laſt Diviſor will be the greateſt Common Meaſure.

EXAMPLE.

If the Numbers be 152, and 184:
Their *Greateſt Common Meaſure* is 8.

```
152)384(2
    304
    ---
     80(152(1
         80
        ---
         72)80(1
            72
           ---
            8)72(9
              72
             ---
              00
```

For

For, Suppose a and b were the Quantities, whose *Greatest Common Measure* is required.

Now suppose $a = 4b + c$.
And $b = 2c + d$.
Also $c = 3d$.

Then $b = 7d$, and $a = 31d$, therefore d is the *Greatest Common Measure* of a and b.

NOTE.

If 1 be the *Greatest Common Measure*, the Numbers are said to be *Prime* to one another.

PROPOSITION 1.

To reduce a Fraction *into its* Least Terms.

RULE.

Divide the Numerator and Denominator by their Greatest Common Measure, their Quotients will be a Fraction equivalent to the former, and in the Least Terms.

Numeral EXAMPLE.

If the given *Fraction* be $\frac{75}{135}$.
The *Greatest Common Measure* is 15.

And $\begin{matrix} 15) \\ 15) \end{matrix} \frac{75}{135} \begin{matrix} (\\ (\end{matrix} \frac{5}{9}$ the given *Fraction* in its *least Terms*.

For $\frac{5}{9} = \frac{75}{135}$ by *Lem.* 2.

And 15 is the *Greatest Number* that will *Divide* 75 and 135 without *Remainder*, by the foregoing *Probl.*

Theref. $\frac{5}{9}$ is the least Term $\frac{75}{135}$ can be brought to.

Note,

NOTE.

When the *Greatest Common Measure* is 1, the *Fraction* is already in the smallest Terms.

For dividing by 1 do's not diminish them.

COROLLARY 1.

A *Fraction* whose *Terms* are *even*, may be abbreviated by a *Continual Division* by 2.

Thus, $\frac{256}{384} = \frac{2}{3}$

For $\begin{matrix} 2) \\ 2) \end{matrix} \frac{256}{384} \Big| \frac{128}{192} \Big| \frac{64}{96} \Big| \frac{32}{48} \Big| \frac{16}{24} \Big| \frac{8}{12} \Big| \frac{4}{6} \Big| \frac{2}{3}$.

COROLLARY 2.

So also may any *Terms*, that are found to be *Divisible* by any other *Digit*.

Thus, $\frac{162}{1296} = \frac{2}{16} = \frac{1}{8}$.

For $\begin{matrix} 3) \\ 3) \end{matrix} \frac{162}{1296} \Big| \frac{54}{432} \Big| \frac{18}{144} \Big| \frac{6}{48} \Big| \frac{2}{16}$.

COROLLARY 3.

When both *Terms* have *Cyphers* adjoyning, cut off equal *Cyphers* from both.

Thus, $\frac{1000}{3000} = \frac{1}{3}$, for $\frac{1\ 000}{3\ 000} = \frac{1}{3}$.

And $\frac{1000}{1400} = \frac{5}{7}$, for $\frac{10\ 00}{14\ 00} = \frac{10}{14} = \frac{5}{7}$.

These *Corollaries* are evident from *Lemma* 2.

Literal

Literal EXAMPLES.

$\frac{xn}{xy}=\frac{n}{y}$; $\frac{25z}{5xz+15az}=\frac{5}{x+3a}$; $\frac{n^2xr^2}{xy^2r}=\frac{n^2r}{yy}$.

This is evident from *Cor.* to *Lem.* 2. *Sect.* 3.

Also $\frac{nnn-naa}{nn+2na+aa}=\frac{nn-na}{n+a}$; $\frac{x^2-y^2}{x^2+2xy+y^2}=\frac{x-y}{x+y}$.

And $\frac{x^6a^2y^2n^2+4x^6a^2n^3m}{y^2m^2r^4+4nm^3r^4}=\frac{x^6a^2n^2}{m^2r^4}$.

PROPOSITION 2.

To Reduce an Integer *into an* Improper Fraction.

CASE I.

When there is no Denominator *assign'd.*

RULE.

Let the given Integer *be a Numerator, and* Unit *as Denominator.*

Thus, $2=\frac{2}{1}$; $15=\frac{15}{1}$; $500=\frac{500}{1}$.

And $n=\frac{n}{1}$; $a+e=\frac{a+e}{1}$.

For *Dividing* by *Unit* does not diminish the *Value.*

N CASE

CASE II.

Where there is a Denominator *assign'd.*

RULE.

Multiply the Integer by the assign'd Denominator, the Product shall be the Numerator.

Numeral EXAMPLE.

If $\left\{ \begin{matrix} 14 \\ 9 \end{matrix} \right\}$ be the given $\left\{ \begin{matrix} \textit{Integer} \\ \textit{Denominator.} \end{matrix} \right.$

Then $14 \left(= \frac{9 \times 14}{9} \right) = \frac{126}{9}$.

For $126 : 9 :: 14 : 1$. by *Def.* of *Divis.*

And $126 : 9 :: \frac{126}{9} : 1$ by *Cor.* 2. *Def.* 1.

Therefore $14 = \frac{126}{9}$.

Or $14 = \frac{14}{1} = \frac{9 \times 14}{9 \times 1} = \frac{9 \times 14}{9} = \frac{126}{9}$ by *Lem.* 2.

Literal EXAMPLES.

$x = \frac{nx}{n}$; $x + y = \frac{xn + yn}{n}$; $a = \frac{am + an}{m + n}$.

For $x \times n = nx$; $\overline{x + y} \times n = xn + yn$, &c.

PROPOSITION 3.

To Reduce mixt Fractions *into* Improper *ones.*

RULE.

RULE.

Multiply the Integer by the Denominator of the Fraction, and add the Numerator to the Product, subscribing the same Denominator.

Numeral *EXAMPLES.*

$$36\tfrac{3}{4}\ \left(= \frac{\overline{36\times4}+3}{4} = \frac{144+3}{4}\right) = \tfrac{147}{4}.$$

For the *Unit* here is consider'd as divided into 4 *Equal Parts.*

Therefore the *Integers* must be *Multiplied* by 4, to produce 4ths.

To which 3 *fourths* being *Added*, the *Sum* will be also 4ths.

Literal *EXAMPLES.*

1. $z + \frac{xn}{a} = \frac{za+xn}{a}.$

2. $a+n-\frac{xx}{ar} = \frac{aar+arn-xx}{ar}.$

PROPOSITION 4.

To Reduce an *Improper Fraction* into an *Integer*, or *Mixt Fraction.*

RULE.

Divide the Numerator by the Denominator, and the Quotient will be the Integer, or mixt Fraction required.

Thus, $\frac{126}{9} = 14$; and $\frac{147}{4} = 36\frac{3}{4}$.

For $\frac{126}{9}$: 1 :: 126 : 9, by *Cor.* 2. *Def.* 1.

And 14 : 1 :: 126 : 9, by *Def.* of *Divis.*

Theref. $\frac{126}{9} = 14$.

Also, $\frac{za + xn}{a} = z + \frac{xn}{a}$; and $\frac{am + an}{m + n} = a$.

PROPOSITION 5.

To Reduce a Fraction *into its* Equivalent *that shall have any assign'd* Denominator.

RULE.

Multiply the Numerator of the Fraction by the assign'd Denominator, and Divide the Product by the Denominator of the Fraction; the Quotient shall be the Numerator required.

EXAMPLE.

To reduce $\frac{3}{4}$ into a *Fraction*, whose *Denominator* is 12.

$$\frac{3}{4} = \frac{4)\,3 \times 12}{12} = \frac{4)\,36}{12} = \frac{9}{12}.$$

For $4 : 3 :: 12 : \frac{3 \times 12}{4} = \frac{36}{4} = 9$.

Theref. $\frac{3}{4}$ $\left(= \frac{4)\,3 \times 12}{12}\right) = \frac{9}{12}$. by *Def.* of *Prop.*

COROL-

COROLLARY.

By this *Proposition*, *Fractions* are reduced into their *Known* Parts of *Time*, *Measure*, *Weight*, *Coin*, &c. As also into *Decimals*, *Sexagessimals*, &c. and the contrary.

Thus, $\frac{3}{4}$ *l.* $=(\frac{3 \times 20}{4}=)$ 15 *s.*

And $\frac{2}{3}$ *Deg.* $=(\frac{2 \times 60}{3}=)$ $40'$

Also $\frac{7}{11}$ *Deg.* $= 38', 10'', 54''', 32''''$, &c.

For 1. $\frac{7}{11}$ *Min.* $=(\frac{7 \times 60}{11}=)$ $38'\frac{2}{11}$.

2. $\frac{2}{11}$ *Min.* $=(\frac{2 \times 60}{11}=)$ $10''\frac{10}{11}$.

3. $\frac{10}{11}$ *Sec.* $=(\frac{10 \times 60}{11}=)$ $54'''\frac{6}{11}$.

4. $\frac{6}{11}$ *Thirds* $=(\frac{6 \times 60}{11}=)$ $32''''\frac{8}{11}$.

&c.

SCHOLIUM.

Hence also, To reduce a *Fraction* into its *Equivalent*, that shall have any *assign'd Numerator.*

EXAMPLE.

To reduce $\frac{3}{4}$ into a *Fraction*, whose *Numerator* is 9.

Then $\frac{3}{4} = (\frac{9}{3) 4 \times 9} =) \frac{9}{12}$.

For $3 : 4 :: 9 : \frac{4 \times 9}{9} = 12$.

PRO-

PROPOSITION 6.

To reduce Fractions *of Different* Denominators, *into their Equivalents, which shall have the same* Denominator.

RULE.

Multiply all the Denominators continually, for a Common Denominator, and each Numerator continually by the other's Denominators, for new Numerators.

Numeral *EXAMPLES.*

1. $\frac{2}{3}$ and $\frac{3}{4}$ make $\frac{8}{12}$ and $\frac{9}{12}$.

$$\left.\begin{array}{l} \text{For } \frac{2}{3} = \frac{2 \times 4}{3 \times 4} = \frac{8}{12} \\ \text{And } \frac{3}{4} = \frac{3 \times 3}{4 \times 3} = \frac{9}{12} \end{array}\right\} \text{by } Lem.\ 2.$$

2. $\frac{1}{2}$, $\frac{3}{4}$, $\frac{5}{7}$, make $\frac{28}{56}$, $\frac{42}{56}$, $\frac{40}{56}$, or $\frac{14}{28}$, $\frac{21}{28}$, $\frac{20}{28}$.

$$\text{For} \left\{\begin{array}{l} \frac{1}{2} = \frac{1 \times 4 \times 7}{2 \times 4 \times 7} = \frac{28}{56} = \frac{14}{28} \\ \frac{3}{4} = \frac{3 \times 2 \times 7}{4 \times 2 \times 7} = \frac{42}{56} = \frac{21}{28} \\ \frac{5}{7} = \frac{5 \times 2 \times 4}{7 \times 2 \times 4} = \frac{40}{56} = \frac{20}{28} \end{array}\right\} \text{by } Lem.\ 2.$$

Literal *EXAMPLES.*

1. $\frac{a}{e}$, $\frac{c}{n}$ make $\frac{an}{en}$, $\frac{ce}{en}$.

2. $\frac{a}{x}$,

2. $\frac{a}{x}, \frac{c}{m}, \frac{rn}{z}$, make $\frac{amz}{xmz}, \frac{cxz}{xmz}, \frac{rnxm}{xmz}$.

$$\text{For} \left\{ \begin{array}{l} \frac{a \times m \times z}{x \times m \times z} = \frac{amz}{xmz} (= \frac{a}{x} \\ \frac{c \times x \times z}{m \times x \times z} = \frac{cxz}{mxz} (= \frac{c}{m} \\ \frac{rn \times x \times m}{z \times x \times m} = \frac{rnxm}{zxm} (= \frac{rn}{z} \end{array} \right\} \text{by } Lem. 2.$$

Therefore they are Equivalent *Fractions*, by *Prop.* 1.

SCHOLIUM.

Hence, to find *Two Integers*, that shall be one to the other as two given *Fractions*.

Suppose the *Fractions* were $\frac{2}{3}$ and $\frac{5}{7}$.

Then $\frac{2}{3} : \frac{5}{7} :: \frac{14}{21} : \frac{15}{21}$ by this *Prop.*
But $\frac{14}{21} : \frac{15}{21} :: 14 : 15$ by *Cor.* 3. *Def.* 1.
Theref. $\frac{2}{7} : \frac{5}{7} :: 14 : 15$ by *Cor.* 16. *Th.* 2. *Ch.* 3.

PROPOSITION 7.

Addition and Subtraction of *Common* Fractions.

Note, If the *Numerators* are not equal, they must be reduc'd to such as have equal ones; then by this

RULE.

The Sum, or Difference of the Numerators set over the Common Denominator, shall be the Sum or Difference of the given Fractions.

Numeral

Numeral *EXAMPLE.*

Let the *Fractions* be $\frac{2}{9}$ and $\frac{5}{9}$.

Then $\frac{5}{9} \pm \frac{2}{9} = \frac{5 \pm 2}{9} = \frac{7}{9}$ or $\frac{3}{9}$.

For $\frac{5}{9} : \frac{2}{9} :: 5 : 2$, by *Cor.* 3. *Def.* 1.

And $\frac{5}{9} \pm \frac{2}{9} : \frac{2}{9} :: 5 \pm 2 : 2$. by *Compound.* or *Divid.*

But $\frac{5 \pm 2}{9} : \frac{2}{9} :: 5 \pm 2 : 2$. by *Cor.* 3. *Def.* 1.

Theref. $\frac{5}{9} \pm \frac{2}{9} : \frac{2}{9} :: \frac{5 \pm 2}{9} : \frac{2}{9}$ by *Cor.* 16. *Theor.* 2.

Conseq. $\frac{5}{9} \pm \frac{2}{9} = \frac{5 \pm 2}{9}$.

Literal Examples in *Addition.*

1. $\frac{aa}{c} + \frac{ee}{c} = \frac{aa + ee}{c}$.

2. $\frac{aa + ax}{a - x} + a = \frac{2aa}{a - x}$.

3. $\frac{aa}{n} + \frac{cc}{m} + \frac{ee}{x} = \frac{a^2 mx + c^2 nx + e^2 nm}{nmx}$.

4. $\frac{125}{x^3 - 25x} + \frac{x - 25}{x^2 + 10x + 25} = \frac{x^3 + 30x^2 + 250x + 625}{x^4 + 5x^3 - 25x^2 - 125x}$

Literal Examples in *Subtraction.*

1. $\frac{aa}{c} - \frac{ee}{c} = \frac{aa - ee}{c}$.

2. $\frac{n^4 + a^3 m}{ncm} - \frac{a^3}{nc} = \frac{nnn}{cm}$.

3. 2 an

3. $\frac{2an+n^2}{a+n} - n = \frac{an}{a+n}$

4. $\frac{a^3+n^3}{cx-xx} - \frac{nnnn}{aac-aax} = \frac{a^5+a^2n^3-n^4x}{a^2cx-a^2x^2}$

The *Reason* is the same with that in *Numbers*.

PROPOSITION 8.

Multiplication *of Common* Fractions.

RULE.

Multiply the { Numerators / Denominators } *for a new* { Numerator. / Denominator.

That is: *Multiply the* Numerators *for a new* Numerator, *and the* Denominators *for a new* Denominator.

Numeral *EXAMPLE.*

$\frac{3}{4}$ by $\frac{2}{3}$ make $\frac{3\times2}{4\times3} = \frac{6}{12} = \frac{3}{6} = \frac{1}{2}$.

For $\frac{4\times2}{4\times3} = \frac{8}{12} = \frac{2}{3}$, by *Lem.* 2.

And $\frac{6}{12} : \frac{8}{12} :: 6 : 8$ by *Cor.* 1. *Def.* 1.

That is, $\frac{6}{12} : \frac{2}{3} :: 3 : 4$

Or, $\frac{6}{12} : \frac{2}{3} :: \frac{3}{4} : 1$

Theref. $\frac{3}{4} \times \frac{2}{3} = \frac{6}{12}$, by the *Nature* of *Proportion*

NOTE.

1. If { *Mixt* / *Compound* } *Fractions*, are to be *Multiplied*, reduce them to { *Improper* / *Single* } ones.

2. If one of the *Factors* be a *whole Number*, it must be made an *Improper Fraction*.

Literal *EXAMPLES.*

1. $\frac{a}{n} \times \frac{e}{m} = \frac{ae}{nm}$

For if $x = \frac{a}{n}$, and $z = \frac{e}{m}$, by *Substit.*

Then $a = xn$, and $e = zm$, by *Mult.*

And $\left(\frac{xnzm}{nm} =\right) \frac{ae}{nm} (= xz) = \frac{a}{n} \times \frac{e}{m}$

2. $\frac{a^2 + 2ab + b^2}{ab} \times \frac{bb}{a+b} = \frac{ab + b^2}{a}$

3. $\frac{an}{m} \times a = \frac{aan}{m}$; and $\frac{aaa}{an + nn} \times a + n = \frac{a^3}{n}$

4. $a + \frac{cc}{a - c} \times a - c = aa - ac + cc$

5. $a + \frac{nn}{a - n} \times a - 2n + \frac{nn}{a} = a^2 - 2an + 2n^2 - \frac{n^3}{a}$

SCHOLIUM 1.

The Product of any Quantity Multiplied by a *Proper Fraction*, is always *Less* than that Quantity.

For in *Multiplying* by an *Unit*, the *Product* will be *equal* to the *Multiplicand*.

But a *Less Multiplier* gives a *Less Product* :

Therefore *Multiplying* by a *Proper Fraction*, (*i. e.* by *Less* than *Unit*) the *Product* must be *Less* than the *Multiplicand*.

SCHOLIUM 2.

Whence the *Product* of two *Proper Fractions* must be *Less* than either of them.

Thus,

Thus, $\frac{2}{3} \times \frac{5}{7} = \frac{10}{21}$, and $\frac{10}{21}$ is *Less* than either $\frac{2}{3}$ or $\frac{5}{7}$.

For $1 : \frac{5}{7} :: \frac{2}{3} : \frac{10}{21}$, by *Def.* of *Mult.*

But $\left.\begin{matrix}\\ \end{matrix}\right\} 1 > \left\{\begin{matrix}\frac{5}{7}\\ \frac{2}{3}\end{matrix}\right\}$ theref. $\left\{\begin{matrix}\frac{2}{3}\\ \frac{5}{7}\end{matrix}\right\} > \frac{10}{21}$.
And

PROPOSITION 9.

Division of *Common* Fractions.

RULE.

Multiply the Dividend by the Divisor's Reverse Fraction; Or, (which is the same) Imagine the Terms of the Divisor changed, then work as in Multiplication.

Numeral *EXAMPLE.*

Thus $\frac{2}{3}) \frac{4}{5} (= \frac{12}{10} = \frac{6}{5} = 1\frac{1}{5}$.

That is, $\frac{3}{2} \times \frac{4}{5} = \frac{12}{10}$.

For $1 : \frac{3}{2} :: \frac{4}{5} : \frac{12}{10}$, by *Def.* of *Mult.*

But $\frac{2}{3} : 1 :: 1 : \frac{3}{2}$, by *Cor.* 4. *Def.* 1.

And $\frac{2}{3} : 1 :: \frac{4}{5} : \frac{12}{10}$, by *Cor.* 15. *Th.* 2. *Ch.* 3. §. 3.

Theref. $\frac{4}{5} \div \frac{2}{3} = \frac{12}{10}$, by the *Nature* of *Proport.*

NOTE.

1. If either *Dividend* or *Divisor* be *whole* or *mixt* Numbers, or if both be *Mixt Numbers*; *Reduce them to improper Fractions.*

2. If they are *Compound Fractions*; *Reduce them to single ones.*

Literal *EXAMPLE.*

1. $\frac{a}{c} \div \frac{x}{n} = \frac{na}{xc}$; and $\frac{ar}{xz} \div \frac{m}{n} = \frac{nar}{mxz}$

2. $\frac{n+3}{5} \div \frac{20a}{3} = \frac{3n+9}{100a}$; and $\frac{aa}{n} \div aa = \frac{1}{n}$

3. $\frac{xxx}{nn} \div a = \frac{xxx}{ann}$; and $a \div \frac{xxx}{nn} = \frac{ann}{xxx}$

4. $a-e \div \frac{a-e}{n} = n$; and $\frac{nnxx}{r} \div \frac{nx}{r} = nx$

5. $\frac{a^2+2an+n^2-ar-nr}{an+ar-nr-rr} \div \frac{a+n-r}{n+r} = \frac{a+n}{a-r}$

Which may be *Demonstrated* thus.

For $\frac{x}{n}, \frac{a}{c} = \frac{xc}{nc}, \frac{an}{nc}$, by *Prop.* 6.

But $\frac{xc}{nc} : \frac{an}{nc} :: xc : an$, by *Cor.* 3. *Def.* 1.

i.e. $\frac{x}{n} : \frac{a}{c} :: xc : an$, by *Equal Division.*

Theref. $\frac{a}{c} \div \frac{x}{n} = \frac{an}{xc}$. by *Def.* of *Proport.*

SCHOLIUM 1.

If the *Fractions* are of one *Denomination*,

Then cast off that Denominator, and Divide the Numerators.

Because *Fractions* having the same *Denominators*, are as their *Numerators*, by *Cor.* 3. *Def.* 1.

SCHO-

SCHOLIUM 2.

The *Quotient* of any *Quantity* divided by a *Proper Fraction* is always *Greater* than that *Quantity*.

For in *Dividing* by *Unit*, the *Quotient* will be equal to the *Dividend* ;

But a *Less Divisor* gives a *Greater Quotient* :

Therefore in *Dividing* by a *Proper Fraction*, (*i. e.* by *Less* than an *Unit*) the *Quotient* must be *Greater* than the *Dividend*.

Or thus, $\frac{2}{3}) \frac{4}{5} (= \frac{6}{5}$, And $\frac{6}{5}$ is *Greater* than $\frac{4}{5}$.

For $1 : \frac{6}{5} :: \frac{2}{3} : \frac{4}{5}$, by *Def.* of *Divis.*

And $1 : \frac{2}{3} :: \frac{6}{5} : \frac{4}{5}$, by *Alternation*.

But $1 > \frac{2}{3}$, theref. $\frac{6}{5} > \frac{4}{5}$.

PROPOSITION 10.

To reduce Compound Fractions *into* Single *ones*.

RULE.

Multiply the { *Numerators* / *Denominators* } *continually for a new* { *Numerator.* / *Denominator.* }

EXAMPLE.

Thus, $\frac{2}{3}$ of $\frac{4}{7}$ being *Reduced*, is $\frac{2 \times 4}{3 \times 7} = \frac{8}{21}$

For if $\frac{4}{7} = n$, then $\frac{2}{3}$ of $\frac{4}{7}$ is $\frac{2}{3}\, n = \frac{2}{3} \times n$

Therefore $\frac{2}{3}$ of $\frac{4}{7} = \frac{2 \times 4}{3 \times 7} = \frac{8}{21}$

DEFINITION IV.

AN Unit *may be imagin'd to be equally divided into* 10 Parts, *and each of those into* 10 *more; so that by a continual Decimal Subdivision, the* Unit *may be supposed to be Divided into* 10, 100, 1000, *&c. equal* Parts, *call'd*, 10*th*, 100*th*, 1000*th*, Parts *of an* Unit.

And since Integers *Increase from* Unit, *towards the Left-hand, in a* Decuple *Proportion, so that a* Figure *in any Place is* Ten *times as much as the same in the next* Place *below it, and but a* Tenth *part of what it signifies in the next* Place *above it; therefore as the* 1*st.* 2*d.* 3*d.* &c. Place *above that of* Units, *is* Tens, Hundreds, Thousands, &c. *So the* 1*st.* 2*d.* 3*d.* &c. Place *below that of* Units, *is* Tenths, Hundredths, Thousandths, &c. *Decreasing in a* Subdecuple *Proportion, as is evident from the following* Table.

Integers.														*Parts.*												
Bill.		*Millions*						*Units*						*Unitth*						*Millionth*						*Bill.*
un.		*th.*			*un.*			*th.*			*un.*			*un.*			*th.*			*un.*			*th.*			*un.*
x	u	c	x	u	c	x	u	c	x	u	c	x	u	x	c	u	x	c	u	x	c	u	x	c	u	x
5	6	7	8	9	9	8	7	6	5	4	3	2	1,	2	3	4	5	6	7	8	9	9	8	7	6	5
&c.																										*&c.*

Whence

Whence those *Fractions*, whose *Denominators* are an *Unit* with a *Cypher* or *Cyphers*, are called *Decimal Fractions*, and may be written without their *Denominators*, distinguished by a *Point* or *Comma* prefix'd, and read like *Integers*, giving them the name of the last *Place* to the *Right-hand*.

Thus, { .5 / .25 } signifies { 5 *Tenths.* / 25 *Hundredths*, }

COROLLARY.

As / So { *Cyphers* before { *Integers* / *Decimals* }, (advancing them so many Places towards the { *Left* / *Right* } *hand*) { *Increase* / *Diminish* } their *value*.

SCHOLIUM.

1. Hence it is, That 1, 2, 3, &c. *Cyphers* before a *Decimal*, advance it so many Places forward, whereby 'tis made 10, 100, 1000, &c. times less. Thus,

.25 ...	signifies	25 *Hundredths*
.025 ..		25 *Thousandths*
.0025 .		25 *Ten Thousandths*
.00025		25 *Hundred Thousandths.*

2. Therefore, a Figure in the 1st. 2d. 3d. &c. *Decimal Place*, is 10, 100, 1000, &c. times less than if it were an *Integer*.

3. Consequently, each removal of a *Figure* into a *Place* forward makes it *Ten* times less than it was before.

PRO-

PROPOSITION II.

To reduce Vulgar Fractions *into* Decimals.

RULE.

To the Numerator add as many Cyphers as you would have Decimal Places; then divide it by the Denominator, and the Quotient (if there be no Remainder) will be a Decimal equivalent to the vulgar Fraction given.

But when there is a Remainder, you may by adding more Cyphers, proceed so as to bring the Quotient nearly equal.

Thus, $\frac{1}{2} = \frac{1.0}{2} = .5$, for 2 : 1 :: 1.0 : .5

$\frac{3}{4} = \frac{3.00}{4} = .75$, for 4 : 3 :: 1.00 : .75

$\frac{2}{7} = \frac{2.000000\text{ \&c.}}{7} = .285714 = \frac{2}{7}$ near, not wanting $\frac{1}{1000000}$ *part* of an *Unit*.

For 7 : 2 : : 1.000000, &c. : .285714, &c.

SCHOLIUM.

'Tis observed when a *Fraction* is reduced to the *smallest Terms*:

That if its *Denominator* be Compounded only of the *Prime Numbers* 2 and 5 (the *Components* of 10) the *Decimal* of that *Fraction* will be determin'd.

But if the *Denominator* be Compounded of any other *Prime Numbers*, it will be Indetermined: and the same Figures will return again in order, and continue to *Circulate*, either by one Figure, or by two, three, &c. Figures, tho' never by more than the Number of *Units* in the *Denominator* less 1.

For

For the *Remainder* being always less than the *Divisor*, therefore may be any Number less by 1 than it:

But in so many Operations, at most, as there are *Units* in the *Divisor*, one of the *Remainders* must return again;

Therefore, the same Figure in the *Quotient* must also return, and so continue the *Circulation*.

To find the Number of the Circulating Figures.

1. Divide the *Denominator* by 2 and 5 as often as possible; if it come to be 9, 99, 999, 9999, &c. or an *Aliquot part* of such Number, or a *Number Compounded* of 2 or 5, and such *Aliquot Part*; *Then the Number of the Circulating Figures, will be equal to so many Figures of 9, as there are in the Number found.*

2. If one of the *Prime Numbers* compounding the *Denominator* (excluding those of 2 and 5.) be not an *Aliquot part* of the other:

Then the Number of the Circulating Figures will be equal to the Product of them required by those Compounding Prime Numbers.

3. And when the *Denominator* is Compounded of 2, or 5, or any Power of them; *Then the Circulating Figures begin at such a Place of Decimals, as is denoted by the Index of 2 or 5, assumed in that Composition, more 1.*

PROPOSITION 12.

Addition *and* Subduction *of* Decimal *Fractions.*

P RULE.

RULE.

Place every Figure under that of the like Name, and Add, *or* Subduct, *as if they were Integers.*

Thus,

To	34.25	From	16.5
Add	3.026	Subd.	.125
Sum	37.276	Rem.	16.375

The *Reason* is the same with that of *Integers.*

PROPOSITION 13.

Multiplication *of* Decimal *Fractions.*

RULE.

Multiply the Factors as if all were Integers; And the Decimals in the Product, must be equal to the Sum of those in both Factors; if they are not, prefix Cyphers to supply the Defect.

For the *Index* of each *Figure* in the *Product*, must be equal to the *Sum* of the *Indices* of the *Multiplied* and *Multiplying Figures.*

Thus,

Mult. 3.52 by 4.3, the *Product* is 15.136

For $\left\{\begin{matrix} 3.52 = \frac{352}{100} \\ 4.3 = \frac{43}{10} \end{matrix}\right\}$ and $\frac{352}{100} \times \frac{43}{10} = \frac{15136}{1000} = 15.136$

Also, .013 *Mult.* by .005, gives .000065

For $\left\{\begin{matrix} .013 = \frac{13}{1000} \\ .005 = \frac{5}{1000} \end{matrix}\right\}$ and $\frac{13}{1000} \times \frac{5}{1000} = \frac{65}{1000000} = .000065$

SCHO-

SCHOLIUM 1.

When a *Decimal*, or *Mix'd Number* is to be *Multiplied* by an *Unit* with *Cyphers*; *'Tis but removing the Point or Comma so many Places towards the Right-hand in the Multiplicand, as there are Cyphers annex'd to the Unit.*

Thus,

.2537 *Mult.* by	10		*Prod.* is	2.537
	100			25.37
	1000			253.7
	10000			2537
	100000			25370

SCHOLIUM 2.

In *Multiplication*, if it were required to find only an assign'd part of the *Product*.

1. *Set the Unit Place of the Lesser Number under that Place of the Greater, whose Index is equal to that of the design'd Right-hand Place of the Product* (i. e. *the first in Integers, and the last in Decimals) or to the Number of Figures to be cut off in the Integers, or left in the Decimals.*

2. *Then set the rest of the Figures of the Lesser Number in a contrary Order.*

3. *And begin to Multiply always at that Figure of the Multiplicand, which stands over the Multiplier.*

Note, *That particular regard must be had of the Increase that would arise from the foregoing Figures of the Multiplicand.*

4. *Set the Right-hand Places of the Products of each Figure in the Multiplier under one another, and their Sum will be the Product required.*

EXAMPLE.

In *Mult.* 1098612286
By 4342945

Product. 4771212734412270

But suppose in this *Product* of 16 *Places*; the first 5 *Places* only were required, then by the foregoing *Rule.*

Indices	11	10	9	8	7	6	5	4	3	2	1	0
Multiplicand	.	.	1	0	9	8	6	1	2	2	8	6
Multiplier	5	4	9	2	4	3	4	.	.	.	.	.
Indices.	0	1	2	3	4	5	6	7	8	9	10	11

43944
3296
439
22
10

47712 required.

$\begin{matrix} 2 & 1 & 0 & \bar{1} & \bar{2} & \bar{3} & \bar{4} \end{matrix}$
In the *Prod.* of 798.0625
$\begin{matrix} 1 & 0 & \bar{1} & \bar{2} \end{matrix}$
By 78.54
Required the *Integers* only.

798.0625
45.87
55864
6384
399
31
62679 Required.

$\begin{matrix} 2 & 1 & 0 & \bar{1} \end{matrix}$
In the *Prod.* of 419.3
$\begin{matrix} 0 & \bar{1} & \bar{2} & \bar{3} & \bar{4} \end{matrix}$
By 0.6375
Reqd. the *Integers* only

419.3
5736.0
251
13
3
267 Required.

$\begin{matrix} 2 & 1 & 0 & \bar{1} & \bar{2} & \bar{3} & \bar{4} \end{matrix}$
In the *Prod.* of 798.0625
$\begin{matrix} 1 & 0 & \bar{1} & \bar{2} \end{matrix}$
By 78.54
Reqd. 2 *Decimal Places.*

798.0625
45.87
5586437
638450
39903
3192
62679.82 Reqd.

$\begin{matrix} 2 & 1 & 0 & \bar{1} & \bar{2} & \bar{3} & \bar{4} \end{matrix}$
In the *Prod.* of 535.5625
$\begin{matrix} \bar{1} & \bar{2} & \bar{3} & \bar{4} \end{matrix}$
By .6375
Reqd. 3 *Decimal Places*

535.5625
5736.0
321338
16067
3749
267
341.421

The *Reason* of this *Contraction* is obvious:

For the *Index* of the *Right-hand Figure* of any *Product* is the *Sum* of the *Indices* of the *Factors*;

And by *inverting* the position of the *Figures* (as the *Rule* directs) the *Sums* of the *Indices* of each Corresponding *Place* in the *Factors*, will be equal among themselves; and therefore equal to the *Index* of the *Right-hand-Place* of the *Product* required:

But

But *Products*, whose *Indices* are equal, belong to the same *Place*, therefore must be set under each other, and their *Sum* must be the *Product* required.

PROPOSITION 14.

Division *of* Decimal *Fractions*.

RULE.

Divide as if all were Integers; (annexing Cyphers to the Dividend, if need be): And

Let the first Figure in the Quotient be of the same Name (i. e. have the same Index) with that Figure of the Dividend, which stands, (or is imagin'd to stand) over the Unit Place of the Divisor.

For the *Index* of each *Figure* in the *Quotient*, must be equal to the *Index* of the *Divided Figure* less by the *Index* of the *Dividing Figure*.

Or, the Decimal Places in the Divisor and Quotient must be equal to those in the Dividend; If they are not prefix Cyphers to the Quotient, to supply the Defect.

For the *Dividend* is equal to the *Product* of the *Divisor* and *Quotient*;

But both *Factors* contain as many *Decimal Places* as the *Product* does:

Therefore, what *Decimal Places* are in the *Dividend* more than in the *Divisor*, must be supplied in the *Quotient*.

Thus,

Thus, .0325 *Divided* by .25, gives .13 in the *Quotient*.

$$\text{For} \begin{cases} .0325 = \frac{325}{10000} \\ .25 = \frac{25}{100} \end{cases} \text{and } \frac{25}{100}) \ \frac{325}{10000} \ (= \frac{32500}{250000} = .13$$

SCHOLIUM.

1. If the *Divisor* be *Greater*, or have more *Decimal* Places, than the *Dividend*; *then by annexing Cyphers to the Dividend, the Quotient may be had to any accuracy.* Thus; .25) .07864, 00 (.31456.

2. Therefore, when there is a *Remainder* after *Division*, (tho' neither *Dividend* or *Divisor* consist of any *Decimals*) *'tis but adding Cyphers to the Dividend, and so proceed to any exactness.*

3. When a Decimal or Mix'd Number is to be Divided by an Unit with Cyphers, *'Tis but removing the Point or Comma in the Dividend, so many Places further towards the Left-hand, as there are Cyphers annex'd to the Unit, prefixing Cyphers to the Dividend to supply vacancy, if need be.* Thus,

$$253.7 \text{ Divided by} \begin{cases} 10 \\ 100 \\ 1000 \\ 10000 \end{cases} \text{Quotient is,} \begin{cases} 25.37 \\ 2.537 \\ .2537 \\ .02537 \end{cases}$$

4. In *Dividing* by an *Infinite Number*, the *Division* may oftentimes be very usefully *Contracted*, by

Taking as many of the Left-hand Figures of the Divisor as you think convenient, for the first Divisor, by which Divide the given Number, and omit one Figure of the Divisor at each following Operation.

Thus,

```
10986, 122|86345) 47712127 (4342,94 —
                  3767639
                   471863
                    32359
                    10387
                      305
                       69
                      &c.
```

PROPOSITION 15.

To reduce a Decimal *into a* Common Fraction.

RULE.

Multiply the Numerator of the Decimal by the Denominator of the Common Fraction.

Thus, .575 of a *l.* is 11 *s.* 6 *d.*

```
      .575   For 1000 : ,575 :: 20s. : 11 s 5/10.
        20
s. 11|.500
        12   And 10 : ,5   :: 12d. : 6 d.
d.  6|.000
```

PROPOSITION 16.

To find a Multiplier *that shall Effect the same as a given* Divisor.

CASE I.

If the Divisor *be an* Integer; *make it a* Denominator, *and* 1 *the* Numerator, *that* Fraction *shall be the* Multiplier *required.*

For

$$\text{For } \frac{N}{D} = N \times \frac{1}{D}.$$

CASE II.

If the *Divisor* be a *Fraction.*

Its Reverse Fraction shall be the Multiplier required.

$$\text{For } \frac{n}{d}) \frac{N}{D} (= \frac{d}{n} \times \frac{N}{D}.$$

NOTE.

Hence, *Divisor* given : 1 :: 1 : *Multiplier* required.

And by finding of a *Factor*, when a *Divisor* is given; or a *Divisor*, when a *Factor* is given; one may advantagiously Contract several tedious Operations in *Arithmetic*, and find variety of excellent useful Rules, for ease and expedition in *Common* and *Practical Accompts.*

CHAP. VI.

The Arithmetic of Incommensurables.

DEFINITION I.

THose Quantities, *which have the same* Common Measure, *are said to be* Commensurable.

SCHOLIUM 1.

Therefore *Commensurable Quantities* may be express'd by *Numbers*; or are as *Number* to *Number.*

DEFINITION II.

AND *those* Quantities *which have not the same* Common Measure, *are said to be* Incommensurable.

SCHOLIUM 1.

Therefore *Incommensurable Quantities* are not as *Number* to *Number*, that is, their Magnitudes cannot be express'd by *Numbers*; yet they may be *Commensurable* in *Power*.

SCHOLIUM 2.

Therefore the *Roots* of *Imperfect Powers* are *Incommensurable Quantities*, and are usually call'd *Surd Roots*, because inexpressible by any known way of *Notation*, otherwise than by their *Radical Signs* or *Indices*.

And tho' *Incommensurable Quantities* have no *Ratio* of *Number* to *Number* with *Commensurable* ones, yet they may be *Commensurable* one to another.

PROPOSITION 1.

To reduce Roots *into their most* Simple Expressions.

RULE.

Divide the given Power by the greatest Power, denoted by the Index, contained therein, that Measures it without Remainder; let the Quotient be affected by the Radical Sign, and have the Root of the Divisor prefix'd; or connected by the Sign ×.

Thus, to reduce $a^{\frac{1}{n}}$ into its *most Simple Expressions* or *Lowest Term*.

Suppose

Suppose x^n the greatest n Power, that will Divide a without Remainder.

Let $y = \frac{a}{x^n}$, then will $a^{\frac{1}{n}} = x \times y^{\frac{1}{n}}$

For $a = x^n y$, theref. $a^{\frac{1}{n}}$ $(= \overline{x^n y}|^{\frac{1}{n}} = \overline{x^n}|^{\frac{1}{n}} \times y^{\frac{1}{n}}) = x \times y^{\frac{1}{n}}$.

So $\overline{a^3y - a^2y^2 + 2a^2yx + ayx^2 - ay^3 + y^2x^2 - 2y^3x + y^4}|^{\frac{1}{2}}$ will be $\overline{a + x - y} \times \overline{ay + y^2}|^{\frac{1}{2}}$.

And, $\overline{a^6 - 9a^5 + 27a^4 - 15a^3 - 108a^2 + 324a - 324}|^{\frac{1}{3}}$ will be $\overline{a - 3} \times \overline{a^3 + 12}|^{\frac{1}{3}}$.

Also $\overline{75}|^{\frac{1}{2}}$ and $\overline{27}|^{\frac{1}{2}}$ will be $5 \times 3^{\frac{1}{2}}$ and $3 \times 3^{\frac{1}{2}}$

For $\frac{75}{25} = \frac{27}{9} = 3$, by the *Rule*

Theref. $\frac{\overline{75}|^{\frac{1}{2}}}{5} = \frac{\overline{27}|^{\frac{1}{2}}}{3} = 3^{\frac{1}{2}}$, by *Evolution*

Then, $\overline{75}|^{\frac{1}{2}} = 5 \times 3^{\frac{1}{2}}$, and $\overline{27}|^{\frac{1}{2}} = 3 \times 3^{\frac{1}{2}}$ by *Mult.*

PROPOSITION 2.

To reduce Roots *of* Different Names, *into those of the same* Name.

RULE.

Involve the Powers Reciprocally according to each others Index, for new Powers;
And let the Product of the Indices, be the Common Index.

Thus, $a^{\frac{1}{n}}$ and $e^{\frac{1}{m}}$ will be $\overline{a^m}|^{\frac{1}{nm}}$ and $\overline{e^n}|^{\frac{1}{nm}}$

$\overline{axx}|^{\frac{1}{n}}$ and $\overline{ezyyy}|^{\frac{1}{m}}$, will be $\overline{\overline{axx}|^{m}}|^{\frac{1}{nm}}$ and $\overline{\overline{ezyyy}|^{n}}|^{\frac{1}{nm}}$

Also $b \times a^{\frac{1}{n}}$ and $c \times e^{\frac{1}{m}}$, will be $b \times \overline{a^m}|^{\frac{1}{nm}}$ and $c \times \overline{e^n}|^{\frac{1}{nm}}$

This will be evident, when 'tis considered

That any *Root*, $\overline{xx}|^{\frac{1}{2}} = \overline{x^3}|^{\frac{1}{3}} = \overline{x^4}|^{\frac{1}{4}} = \overline{x^5}|^{\frac{1}{5}}$, &c.

COROLLARY 1.

Hence, *Rational Quantities* may be reduced to the *Form* of any assign'd *Root*.

Thus, a reduced to the *Form* of $x^{\frac{1}{n}}$ is $\overline{a^n}|^{\frac{1}{n}}$

COROLLARY 2.

Also, *Roots* with *Rational Coefficients* may hence be reduced so as to be wholly affected by the *Radical Sign*.

Thus, $a \times e^{\frac{1}{n}} = \overline{a^n e}|^{\frac{1}{n}}$, and $ay^2 \times \overline{x^3 r}|^{\frac{1}{n}} = \overline{x^3\, r\, \overline{ayy}|^{n}}|^{\frac{1}{n}}$

COROLLARY 3.

Hence is known, whether any two given *Roots* are *Commensurable* one to another; As also, to find their *Ratio*.

For after Reduction into the lowest Terms, and the same Name;

If the Powers are equal, The Roots are Commensurable; And their Ratio is equal to that of the Rational Coefficients.

Thus,

Thus, $\sqrt{75}$ and $\sqrt{27}$ reduc'd, will be $5\sqrt{3}$ and $3\sqrt{3}$ which are *Commensurable*.

For $5\sqrt{3} : 3\sqrt{3} :: 5 : 3$, by *Cor.* 5. *Th.* 2. *Ch.* 3.

PROPOSITION 3.

Multiplication *and* Division *of Simple* Roots.

RULE.

Let the Product, *or* Quotient, *of the* Powers *be affected by the Radical Sign, prefixing the* Product, *or* Quotient, *of the Rational Coefficients (if there be any).*

Examples in Multiplication.

Thus, $a^{\frac{1}{n}} \times e^{\frac{1}{n}} = \overline{ae}|^{\frac{1}{n}}$; and $\overline{ar}|^{\frac{1}{n}} \times \overline{ex}|^{\frac{1}{n}} = \overline{aerx}|^{\frac{1}{n}}$

Also, $a^{\frac{1}{n}} \times e = \overline{ae^{n}}|^{\frac{1}{n}}$; and $a^{\frac{1}{n}} \times e^{\frac{1}{m}} = \overline{a^{m}e^{n}}|^{\frac{1}{nm}}$

For $1 : a :: e : ae$, by *Def.* of *Multiplication*.

But $1^{\frac{1}{n}} : a^{\frac{1}{n}} :: e^{\frac{1}{n}} : \overline{ae}|^{\frac{1}{n}}$, by *Cor.* 4. *Theor.* 2. *Ch.* 3.

Theref. $a^{\frac{1}{n}} \times e^{\frac{1}{n}} = \overline{ae}|^{\frac{1}{n}}$, by *Conv. Theor.* 2. *Ch.* 3.

Examples in Division.

Thus, $a^{\frac{1}{n}} \div e^{\frac{1}{n}} = \overline{\frac{a}{e}}|^{\frac{1}{n}}$, and $ba^{\frac{1}{n}} \div ca^{\frac{1}{n}} = \frac{b}{c}$

Also $a \div e^{\frac{1}{n}} = \overline{\frac{a^{n}}{e}}|^{\frac{1}{n}}$, and $a^{\frac{1}{n}} \div e^{\frac{1}{m}} = \overline{\frac{a^{m}}{e}}|^{\frac{1}{nm}}$

For

For $e : 1 :: a : \frac{a}{e}$ by *Def.* of *Division.*

But $e^{\frac{1}{n}} : 1^{\frac{1}{n}} :: a^{\frac{1}{n}} : \overline{\frac{a}{e}}|^{\frac{1}{n}}$ by *Cor.* 4. *Th.* 2. *Ch.* 3.

Theref. $a^{\frac{1}{n}} \div e^{\frac{1}{n}} = \overline{\frac{a}{e}}|^{\frac{1}{n}}$ by *Conv. Th.* 2, *Ch.* 3.

NOTE.

If the *Powers* be the same; *Then the Power affected by the Sum, or Difference of the Indices, shall be the Product, or Quotient sought.*

In MULTIPLICATION.

1. $y^{\frac{n}{m}} \times y^{\frac{r}{s}} = y^{\frac{ns+mr}{ms}}$; and the $\frac{n}{m}$ Power of $y^{\pm\frac{r}{s}}$ is $y^{\pm\frac{nr}{ms}}$

2. $\frac{1}{y} \times \frac{1}{\overline{y^{n}}|^{\frac{1}{m}}}$, or $y^{-1} \times y^{-\frac{n}{m}} = y^{-\frac{m+n}{m}} = \frac{1}{\overline{y^{m+n}}|^{\frac{1}{m}}}$

3. $\frac{1}{y^{\frac{n}{m}}} \times \frac{1}{y^{\frac{r}{s}}}$, or $y^{-\frac{n}{m}} \times y^{-\frac{r}{s}}$ will be

$y^{-\frac{sn+rm}{sm}} = \frac{1}{y^{\frac{sn+rm}{sm}}} = \frac{1}{\overline{y^{sn+rm}}|^{\frac{1}{sm}}}$; Also

4. $\frac{1}{\overline{ay^{m}}|^{\frac{1}{n}}} \times \frac{1}{\overline{y^{r}}|^{\frac{1}{s}}}$, or $a^{-\frac{m}{n}} \times y^{-\frac{m}{n}} \times y^{-\frac{r}{s}}$ will be

$a^{-\frac{m}{n}} \times y^{-\frac{sm+rn}{sn}} = \frac{1}{a^{\frac{m}{n}} \times y^{\frac{sm+rn}{sn}}}$, or

$\frac{1}{\overline{a^{n}}|^{\frac{1}{m}} \times \overline{y^{sm+rn}}|^{\frac{1}{sn}}}$

In

In *DIVISION.*

1. $y^{\frac{n}{m}} \div y^{\frac{r}{s}}$ will give $y^{\frac{sn-rm}{sm}}$

And $y^{-\frac{1}{n}} \div y^{\frac{1}{m}} = y^{-\frac{1}{n}-\frac{1}{m}} = y^{-\frac{m+n}{mn}}$

2. $\frac{1}{\overline{y^{n}}|^{\frac{1}{m}}} \div \frac{1}{y}$ or $y^{-\frac{n}{m}} \div y^{-1} = y^{-\frac{n-m}{m}} = \frac{1}{\overline{y^{n-m}}|^{\frac{1}{m}}}$

PROPOSITION 4.

Addition *and* Subtraction *of like Simple* Roots.

General RULE.

Let the given Roots be Connected by the Sign $+$, *or* $-$.

1. But if the *Roots* are *Commensurable*; then,

Since in any two Quantities, a and e, $\overline{\frac{a}{e} \pm 1} \times e = a \pm e$.

Therefore,

$$\left\{\begin{array}{l} 8^{\frac{1}{2}} + 2^{\frac{1}{2}} = \overline{18}|^{\frac{1}{2}} \\ \overline{18}|^{\frac{1}{2}} - 2^{\frac{1}{2}} = 8^{\frac{1}{2}} \end{array}\right\} \text{and} \left\{\begin{array}{l} \overline{768}|^{\frac{1}{4}} + 3^{\frac{1}{4}} = \overline{1875}|^{\frac{1}{4}} \\ \overline{1875}|^{\frac{1}{4}} - 3^{\frac{1}{4}} = \overline{768}|^{\frac{1}{4}} \end{array}\right.$$

2. And if the *Roots* are reduced into their *lowest Terms*;

Then the Sum, *or* Difference, *of the Rational Coefficients Multiplied by the Common Root, will be the* Sum, *or* Difference, *of the given Roots.*

Thus, $a \times r^{\frac{1}{n}} \pm b \times r^{\frac{1}{n}} = \overline{a \pm b} \times r^{\frac{1}{n}}$

For, if $r^{\frac{1}{n}} = c$, then $a \times r^{\frac{1}{n}} = ac$, and $b \times r^{\frac{1}{n}} = bc$, theref.

$ac \pm bc = \overline{a \pm b} \times c = \overline{a \pm b} \times r^{\frac{1}{n}} = a \times r^{\frac{1}{n}} \pm b \times r^{\frac{1}{n}}$

3. Also

3. Also the *Sum*, or *Difference*, of any two *Square Roots* is equal to

The Square Root of the Sum, or Difference, between the Sum of the Powers, and twice the Product of their Roots.

Thus, $\sqrt{75} + \sqrt{48} = \sqrt{243}$ or $9\sqrt{3}$
Also $\sqrt{9} + \sqrt{4} = \sqrt{25} = 5$
For suppose $\sqrt{aa}$, and $\sqrt{ee}$ any given Quantities,
'Tis plain, $\overline{aa}|^{\frac{1}{2}} \pm \overline{ee}|^{\frac{1}{2}} = a \pm e = \overline{aa \pm 2ae + ee}|^{\frac{1}{2}}$

NOTE.

The *Operations* of *Compound Surds* are so easily deduced from what has been said of *Simple ones*, that we need not insist on them.

CHAP. VII.

The Method of Resolving Mathematical Problems.

General Directions.

I. *It is absolutely necessary to have a clear and distinct Conception of the* Conditions *of the* Question *propos'd to be* Resolv'd.

II. *Then substitute some* Character *or* Letter, *for each* Quantity *concern'd in the* Question.

And because in *Algebraic* Operations, the *known* and *unknown Quantities* are oftentimes very much Complicated; 'tis usual, for easier distinction, to put the *first Letters* of the Alphabet, for *known*, and the *latter* for *unknown Quantities.*

But

But if *Quantities* of different *Species* be *Noted* by the *first* Letter of their *Names* (when it can conveniently be done) the *Expression* will then be so Simple, so readily known and remember'd, as to be of vast help and advantage to the *Imagination*; besides, it shortens the *Process*, yet preserves it free from all obscurity; and at the same time, wonderfully facilitates the *Solution* it self; as the *Ingenious Learner* will soon find by *Practice*.

III. *By due reasoning from the* Conditions *of the* Question, *let the* Quantities *concerned therein be justly* Stated, *and carefully compared*; *so that their* relation *to one another may appear*, *and the* difference *which renders them* unequal, *be discovered*; *and consequently the same* Thing *found expressible two ways or brought into an* Equation, *or into several* Equations *independent on each other.*

NOTE.

1. *If there are as many Equations given, as required.*

Then the Question has a determinable Number of *Solutions*;

Because, each Quantity therein has but one value.

As suppose a *Question* proposed concerning the Age of 3 *Persons*, was thus Condition'd, *viz.*

The 2d. is 7 Years older than the 1st. The Age of the 3d. is Triple that of the 1st. and 2d. And the Sum of their Age is 68 Years; required the Age of each?

Put z for the *Age* of the 1st.

Then z, $\overline{z+7}$, $6z+21$ is the *Age* of the 1st. 2d. and 3d.

And $z+\overline{z+7}+\overline{6z+21}$, or $8z+28=68$, by the conditions of the *Question*.

So that here is but one *Equation given*, and one *required*.

2. *If there are more Equations given than required.*

Then the Excesses are superfluous, and may sometimes happen to be contrary to, or inconsistent with the rest; and by that means, not only limit, but render the *Question* incapable of any *Solution*.

3. *If there are more Equations required than given.*

Then the *Question* is *Imperfectly* determined;

Because capable of an *Infinite number* of *Answers*.

As suppose a *Question* propos'd concerning the Age of three *Persons*, was thus condition'd, *viz.*

The Age of the first is equal to the square of the Age of the second;

And the Sum of the Age of the 1st. and 2d. is equal the Age of the 3d.

Required the Age of each.

Put x, y, and z, for the *Age* of the 1st. 2d. and 3d. *Person*.

Then $x = yy$, and $x + y = z$ by the *Conditions* of the *Question*.

So that here are two *Equations* given, and three required:

Therefore the *Question* is *Imperfectly determin'd*;

For any one of these Quantities may be assumed at Pleasure.

IV. If the *Question* when *stated*, is found to have a determinable Number of *Solutions*:

Then the *Equation* directly drawn from the *Conditions* of the *Question*, (because at that time the *known* and *unknown Quantities*, are for the most part very much Compounded and mix'd together on each *side* thereof) must be reduc'd into another, by *Equal Augmentation and Diminution*,

where

where { only *one Power* / *different Powers* } of the *unknown Quantity* { is / are } found equal to some *known* ones; which is call'd a { *Simple* or *Pure* / *Compound* or *Affected* } *Equation*; And is then prepared for *Solution*, and distinguish'd by the *Name* of the *Index* of the *Highest Power* of the *Quantity sought.*

V. Let the *Equation* be cleared from all *Fractions* and *Surd Roots*, as also the *highest unknown Power* from any *Coefficient*.

VI. Let all the *Terms* on the *Right-side* be *Transpos'd* to the *Left*, so that the whole be made equal to *nothing*; And let the several *like Terms* be brought into one, by prefixing their *Signs* and *Coefficients*; placing the *highest unknown Power* first, the rest succeeding in Order, and the *Absolute known Quantity* last.

REDUCTION of EQUATIONS.

I. By *Transposition*.

That is, by *Equal Addition*, if the Quantity be *Negative*, and by *Equal Subduction*, if *Affirmative*;

And is performed *by transferring the Quantity to the other side of the Equation, with a contrary sign.*

If	$x - 10 = 40$	If	$x + 10 = 40$
Add	$+10 \quad +10$	Sub.	$-10 \quad -10$
Then	$x - 10 + 10 = 40 + 10$		$x + 10 - 10 = 40 - 10$
i. e.	$x = 50$	Or	$x = 30$

$$\text{If } z^3 - 3az^2 = b^3 - b^2 z + 2az^2$$
$$-2az^2 + b^2 z \qquad + b^2 z - 2az^2$$
$$\text{Then } z^3 - 5az^2 + b^2 z = b^3 \quad 0 \quad 0$$

The *Reason* of this Operation is plain from *Axiom* 1.

II. By

II. *By Equal Multiplication.*

If $\frac{z}{a}=b$. If $\frac{z^3+3a^2}{c}+n+a=z+a$.

Mult. by a a *Mult.* by c c

Then $z=ab$. Then $z^3+3a^2+cn=cz$.

If $\frac{z^3}{4}+az^2-\frac{cz^2}{a}=ac^3$

Then $az^3+4a^2z^2-4cz^2=4a^2c^3$.

The Reason is evident from *Axiom* 2.

And by this means *Equations* are cleared from *Fractions*.

III. By *Equal Division.*

If $ax=c$	$az+ez=cb$	$5y+20=60$
Divid. by a a	$a+e$ $a+e$	5 5
Then $x=\frac{c}{a}$	$z=\frac{cb}{a+e}$	$y=\frac{60-20}{5}=8$

If $z^4+2az^3=b^3z^2$, then $z^2+2az=b^3$.

If $z^4-2az^2=b^3z$, then $z^3-2az=b^3$.

The *Reason* of this Operation is also evident from *Axiom* 2.

VI. By *Equal Involution.*

By means of which *Equations* are *clear'd* from *surd Quantities.*

1. If $\overline{ax}|^{\frac{1}{2}}-b=c$, or $\overline{ax}|^{\frac{1}{2}}=c-b$.

Then $ax=c^2-2cb+b^2$, by *Squaring each side*,

2. If

2. If $x - a = \overline{abx - a^2 c}|^{\frac{1}{3}}$; Then,
$x^3 - 3ax^2 + 3a^2 x - a^3 = abx - a^2 c$, by *Cubing*.
3. If $x^3 * - x\sqrt{a} + \sqrt{b} = 0$, put $\sqrt{a} = m$, and $\sqrt{b} = n$

Then $x^3 * - mx + n = 0$. And $m = \frac{x^3 + n}{x}$ by *Tr.* & *Di.*

Theref. $m^2 x^2 = x^6 + 2nx^3 + n^2$, by *Mult.* and *Invol.*

And $-n = \frac{x^6 - m^2 x^2 + n^2}{2x^3}$ by *Transf.* and *Divis.*

Therefore, $x^{12} * - 2m^2 x^8 - 2n^2 x^6 + m^4 x^4 - 2m^2 n^2 x^2 + n^4 = 0$. by *Mult. Invol.* and *Transf.*

V. By *Equal Evolution*.

1. If $zz = 25$, then $z = \sqrt{25} = 5$.
2. If $x^2 + 2xa + a^2 = c^2$, then $x + a = c$
3. If $z^4 + 2az^3 + a^2 z^2 = m^4 n^2$, Then,
   $z^2 + az = m^2 n$, by *Extr.* the *Sq. Root.*
4. If $z^6 - 3az^5 + 3a^2 z^4 - a^3 z^3 = 8m^3 n^3$,
   Then $z^2 - az = 2mn$, by *Extr.* the *Cube Root.*
5. If $x^2 + 2xa = c^2$, then by *Extr.* the *Sq. R.* of the *unknown side*, we have $x + a$, with $-a^2$ remaining;

Therefore, $x + a \; (= \overline{x^2 + 2xa + a^2}|^{\frac{1}{2}}) = \overline{c^2 + a^2}|^{\frac{1}{2}}$.

And if $x^2 - 4ax = c^2$, then the *Sq. R.* of the *unknown side* will be $x - 2a$, with $-4a^2$ remaining; therefore,

$x - 2a \; (= \overline{x^2 - 4ax + 4a^2}|^{\frac{1}{2}}) = \overline{c^2 + 4a^2}|^{\frac{1}{2}}$.

Therefore this Operation may be performed, by *Adding the Square of half the Coefficient to each side of such Equation*; and is called (not improperly) *Compleating the Square*

6. If

6. If $y^4 + 4ay^3 + 4a^2y^2 = c^2y^2 - 2ac^2y = 2a^2c^2$, then the *Sq. Root* of the *unknown side* will be $y^2 + 2ay - \frac{1}{2}c^2$, with $-\frac{1}{4}c^4$ remaining; therefore,

$$\overline{y^4 + 4ay^3 + 4a^2y^2 - c^2y^2 - 2ac^2y + \tfrac{1}{4}c^4}|^{\frac{1}{2}}$$

will be equal to $\overline{2a^2c^2 + \frac{1}{4}c^4}|^{\frac{1}{2}}$

That is $y^2 + 2ay - \frac{1}{2}c^2 = \overline{2a^2c^2 + \frac{1}{4}c^4}|^{\frac{1}{2}}$

SCHOLIUM.

Whence the Ten first *Propositions* of the second Book of *Euclid* appear at Sight:

By putting $s = a + e$, then is $a = s - e$, and $e = s - a$.

PROPOSITIONS.

1. $ns = na + ne$ } by *Equal Multiplication.*
2. $ss = sa + se$ }
3. $sa = a^2 + ae$, and $se = e^2 + ae$ by *Eq. Mult.*
4. $ss = a^2 + 2ae + e^2$, by *Equal Involution.*
5. $\frac{1}{4}ss = \frac{1}{4}d^2 + ae$, for $\frac{1}{4}s^2 - \frac{1}{4}d^2 = ae$.
6. $sd + e^2 = a^2$, for $sd = a^2 - e^2$.
7. $\left\{\begin{array}{l} 2sa + e^2 = s^2 + a^2 \\ 2se + a^2 = s^2 + e^2 \end{array}\right\}$ for $\left\{\begin{array}{l} e^2 = s^2 - 2sa + a^2 \\ a^2 = s^2 - 2se + e^2 \end{array}\right.$
8. $\left\{\begin{array}{l} \overline{s+a}|^2 = s^2 + 2sa + a^2 = 4sa + e^2 \\ \overline{s+e}|^2 = s^2 + 2se + e^2 = 4se + a^2 \end{array}\right\}$ by *Preced.*
9. $\frac{1}{2}s^2 + \frac{1}{2}d^2 = a^2 + e^2$, for $\overline{a+e}|^2 + \overline{a-e}|^2 = 2a^2 + 2e^2$
10. $s^2 + d^2 = 2a^2 + 2e^2$, by *Preceding.*

And by the like Comparisons most of those *Propositions* given by *Vieta*, *Oughtred*, and others, with innumerable more of this kind, of excellent use in the various Parts of the *Mathematicks*, are deduced with great Facility.

The

The Derivation and Composition of EQUATIONS.

Since any *Quantity* compos'd of *Parts* may be reduc'd into *Parts*, it must evidently follow, That the *Original Components* or *Roots* of all *Equations*, may be either *Affirmative*, *Negative*, *Mix'd*, or *Imaginary*.

And because in all *prepared Equations*, any one of its *constitutive Roots* may be put for the *unknown Quantity*;

Therefore, all *prepared Equations* are *Really*, or *Imaginarily* constituted by the *Products* of so many *Affirmative Roots* with *Negative Signs*, or *Negative Roots* with *Affirmative Signs*, connected to the *unknown Quantity*, as is denoted by the *Index* of its *Highest Power*: thus in

The *Origination* of *Quadratic Equations*.

1. If $\left\{\begin{array}{l} x = +a \\ x = +e \end{array}\right\}$ then $\left\{\begin{array}{l} x - a = 0 \\ x - e = 0 \end{array}\right.$

Therefore $\begin{array}{l} xx - ax + ae = 0 \\ \quad\;\; - ex \end{array}$

Put $s = a + e$, then $x^2 - sx + ae = 0$.

2. If $\left\{\begin{array}{l} x = -a \\ x = -e \end{array}\right\}$ then $\left\{\begin{array}{l} x + a = 0 \\ x + e = 0 \end{array}\right.$

Therefore $\begin{array}{l} x^2 + ax + ae = 0 \\ \quad\;\; + ex \end{array}$

Put $s = a + e$, then $x^2 + sx + ae = 0$.

3. If

3. If $\left\{\begin{matrix} x = +a \\ x = -e \end{matrix}\right\}$ then $\left\{\begin{matrix} x - a = 0 \\ x + e = 0 \end{matrix}\right\}$

Therefore $\begin{matrix} xx - ax - ae = 0 \\ +ex \end{matrix}$

Put $d = a \backsim e$; if $+a$ be $\left\{\begin{matrix} \text{greater} \\ \text{less} \end{matrix}\right\}$ than $-e$.

Then $xx \mp dx - ae = 0$

Therefore, all *Quadratic Equations* are reducible to one of these *Forms*.

1. $xx \mp sx + ae = 0$
2. $xx \mp dx - ae = 0$

The *Origination* of any *Cubic*, *Biquadratic*, or higher *Equation*, may after the same manner be easily *deriv'd*.

From which *Comparison* 'tis evident that:

I. In every *Prepared Equation Really* constituted, which has, or is supposed to have, all its *Terms*;

The unknown Quantity has so many Roots or Values, as are the Dimensions of its highest Power; whereof so many are Affirmative as the Signs in Order have Changes, and the rest are Negative.

II. In such *Prepared Equations*, by changing the *Signs* in *even Places*;

1. *The Affirmative Roots are made Negative, and the Negative Roots Affirmative.*

Also, the Coefficient of the 2d. Term is the Sum of all the Roots so sign'd.

And the Coefficient of the 3d, 4th, 5th, *&c. Term, is the Sum of the Products made of the Roots taken by* 2's, 3's, 4's, &c.

But

But the Number of Products, whose Sum is to be taken in the 3d, 4th, 5th, &c. Term is the unciæ of the 3d, 4th, 5th, &c. Term of a Binomial rais'd to the Dimension of this Equation.

Therefore the Last Term or Absolute known Quantity is only the Product of all the Roots.

2. *Hence, If all the Negative Products made of the Roots taken by 2's, 3's, 4's, &c. (Secluding their Signs), are equal to all the Affirmative ones, (tho' not respectively one to another) then the 3d, 4th, 5th, &c. Term is wanting.*

And the contrary, If these are wanting, those must be equal.

PROBLEM I.

To Increase, or Diminish the value of the unknown Roots of an Equation, by any given Quantity.

RULE.

Instead of the unknown Root, Substitute another Less, or Greater than it, by the Quantity given.

Exam. 1. To *Augment* the *Roots* of this *Equation*; $xx + ax - bb = 0$, by the Quantity c.

Put $x = z - c$, then you will have this *Equation* $zz + az - 2cz - ac + cc - bb = 0$, whose *Roots* *exceed* those of the former, by the *Quantity* c.

Exam. 2. To *Diminish* the *Roots* of this *Equation*; $x^2 + ax - bb = 0$, by the Quantity c.

Put $x = z + c$, then you will have this *Equation* $z^2 + az + 2cz + ac + cc - bb = 0$, whose *Roots* are *less* than those of the *Equation* propos'd, by the *Quantity* c.

COROLLARY 1.

To take away any one *Term*, except the *First*, from any *given Equation*.

Let the *Equation* be $x^4 - ax^3 + bx^2 - cx + d = 0$.

Put $y + n = x$. Then

$$\begin{array}{rrrrrl}
y^4 & +4ny^3 & +6n^2y^2 & +4n^3y & +n^4 & = +xxxx \\
 & -a & -3an & -3an^2 & -an^3 & = -axxx \\
 & & +b & +2bn & +bn^2 & = +bxx \\
 & & & -c & -cn & = -cx \\
 & & & & +d & = +d
\end{array}$$

Now 'tis plain, That any *Term*, except the *First*, may be *taken away* from this *Equation*, because n was taken at Pleasure.

So that by putting $4n - a = 0$, or $n = \frac{1}{4}a$, the 2d *Term* must vanish;

And putting $6n^2 - 3an + b = 0$, the 3d *Term* will also vanish:

After the same manner any other *Term* in this, or any other *Equation* may be destroy'd.

Therefore, in *taking away the 2d Term* of any *Equation*; *Let the Index of the highest unknown Power be* m, *and if the Coefficient of the 2d Term be* $\pm a$, *substitute* $y \mp \frac{a}{m}$ *in the room of* x *every where, then the 2d Term of that Equation will be destroy'd.*

COROLLARY 2.

To add to an *Equation* any *Term* that is *wanting*.

Suppose the *Equation* was $x^4 + c^3x - d^4 = 0$

Put

Put $x = y - c$, then,

$$\begin{aligned} y^4 - 4cy^3 + 6c^2y^2 - 4c^3y + c^4 &= + xxxx \\ + c^3y - c^4 &= + c^3x \\ -d^4 &= -d^4 \end{aligned}$$

$$y^4 - 4cy^3 + 6c^2y^2 - 3c^3y - d^4 = 0.$$

PROBLEM 2.

To Multiply, *or* Divide *the Roots of an* Equation, *by any given* Quantity.

RULE.

Multiply, or Divide each Term respectively, by a Rank of Continual Proportionals from 1, *whose Ratio is the Quantity given.*

Exam. 1. To *Multiply* the *Roots of this Equation,*

$$x^3 + ax^2 - b^2x = ccc, \text{ by } 4;$$

Mult. by 1 4 16 64 then,

We have $x^3 + 4ax^2 - 16b^2x = 64c^3$, an *Equation* whose *Roots* are *Quadruple* those of the *former.*

COROLLARY 1.

Hence, an *Equation* may be clear'd from *Fractions*; *By Multiplying its Roots by the Product of the Denominators of the Fraction.*

Thus, if $x^3 - \frac{2}{3}ax^2 + \frac{3}{4}bbx - ccc = 0.$

Mult. by 1 12 144 1728

Product, $x^3 - 8ax^2 + 108bbx - 1728c^3 = 0.$

NOTE.

If the Coefficient *of the* 2d Term *of any* Equation *be not the* Index *of that* Equation's Dimension, *or some* Multiple *thereof, the* Solution *will be incumbred with*

 Fractions;

Fractions; *to avoid which*, Multiply all the Coefficients by a Rank of Continual Proportionals from 1, whose Ratio is the Equation's Dimension.

COROLLARY 2.

Also, *Equations* may by this Rule, sometimes be clear'd from *Surd Quantities*; *By Multiplying the Roots of the Equation by the Surd Quantities to be clear'd.*

Thus, if $x^4 + 2ax^3\sqrt[2]{2} + 8bbx^2 - c^3x\sqrt[2]{8} = 2d^4$

Mult. by 1 $\sqrt[2]{2}$ 2 $\sqrt[2]{8}$ 4

Prod. $x^4 + 4ax^3 + 16bbx^2 - 8c^3x = 8d^4$

Exam. 2. To *Divide* the *Roots* of this *Equation* $x^3 + 6ax^2 - 18bbx - 162c^3 = 0$, by 3;

Divide by 1 3 9 27 then,

We have $x^3 + 2ax^2 - 2b^2x - 6c^3 = 0$, an *Equation*, whose *Roots* are *Thirds* of those of the *former*.

COROLLARY.

Hence also *Equations* are sometimes clear'd from *Surd Quantities*, without being rais'd to higher *Powers*; *By Dividing their Roots by the Surd Quantity to be clear'd.*

Thus, if $x^4 - ax^3\sqrt[2]{2} + 2bbx^2 - cx\sqrt[2]{8} = 2d^4$

Divide by 1 $\sqrt[2]{2}$ 2 $\sqrt[2]{8}$ 4

Quotient $x^4 - ax^3 + bbx^2 - cx = \frac{1}{2}d^4$

And

And if $x^3 - axx\sqrt[3]{2} + bbx\sqrt[3]{32} = cd^2$

Divide by $1 \qquad \sqrt[3]{2} \qquad \sqrt[3]{4} \qquad 2$

Quotient $x^3 - axx + 2bbx = \frac{1}{2}cdd.$

THE SOLUTION OF EQUATIONS.

I. Of *Simple* or *Pure Equations*.

That the Reader might know the manner of applying the foregoing *Rules*, we thought it neceſſary in this Place, to inſert ſome of the eaſieſt Queſtions we could find.

And if the Method of Solution here uſed be well obſerved, the young Learner will find no difficulty in reſolving thoſe that are more intricate.

1. Queſt. *One being ask'd how old he was, Anſwer'd, If* $\frac{1}{20}$ *of my Age be Multiplied by* $\frac{5}{8}$ *of the ſame, the Product will be my Age; I demand his Age.* Suppoſe it x.

Then $\frac{x}{20} \times \frac{5x}{8} = \frac{5xx}{160} = x$, by the *Queſtion*;
And $160x = 5xx$, by *Multiplication*:
Therefore, $160 = 5x$, and $x = \frac{160}{5} = 32$, by *Diviſ.*

Queſt. 2. *One ask'd a Shepherd to ſell him a* 1000 *Sheep, who Anſwered, that he could not then, for he wanted of his Demands juſt as many, half as many, and* $72\frac{1}{2}$ *Sheep more than he had; How many had he?* Suppoſe it x,

Then

Then $x + x + \frac{1}{2}x + 72\frac{1}{2} = 1000$, by the *Question*;
And $2x + \frac{1}{2}x = 1000 - 72,5 = 927,5$, by *Transp.*

Theref. $5x = 1855$, and $x = \frac{1855}{5} = 371$, *sought.*

Quest. 3. *A Privateer running at the rate of* 10 *Miles an Hour discovers a Ship 6 Leagues off, making away at the rate of* 8 *Miles an Hour; I demand in how many hours can the Privateer come in with the Ship.* Suppose in x hours.

Then $8x + 18 = 10x$, by the *Question*;
And $10x - 8x = 18$, by *Transposition*:

Theref. $x = \frac{18}{10-8} = \frac{18}{2} = 9$ *hours* sought.

Quest. 4. *The Age of two Persons* A *and* B, *being* 100 *Years; the Age of* A *exceeds that of* B, *by* 40 *Years: I demand the Age of each.* Suppose them x and y.

$$\text{Then, } \left\{ \begin{array}{l} x + y = 100 \therefore x = 100 - y \\ x - y = 40 \therefore x = 40 + y \end{array} \right\} \therefore y = 30, \text{ and } x = 70.$$

Or thus,

Suppose x = Age of *A*, then $x - 40$ = Age of *B*,
And $2x - 40 = 100$, by the *Question*, theref. $2x = 140$,

Conseq. $x = \frac{140}{2} = 70$, and the Age of *B* is 30.

Quest.

Quest. 5. *The Persons* A, B, C, *owe me Money, but I forgot both the Sum and Particulars, yet by comparing some Accounts I have by me, it appears that the Debt of* A *and* B *is* 47*l.* *of* A *and* C, 71 *l.* *of* B *and* C, 88*l.* *I demand each Man's Debt.* Suppose them x, y, and z.

Then $\begin{cases} x + y = 47 \therefore x = 47 - y \\ x + z = 71 \therefore x = 71 - z \\ y + z = 88 \therefore y = 88 - z = z - 24 \end{cases}$ $\therefore y = z - 24.$

Theref. $z = 56l.$ $y = 32\,l.$ and $x = 15\,l.$

Quest. 6. *A Gentleman bought a House, a Park, and a Garden; their prices were as* 12, 5, *and* 1; *but the Sum of double the price of the House, triple the price of the Park, and quadruple the price of the Garden, is so much greater than* 10000 *Pounds, as the Sum of the price of the House and Park is less than* 5000 *Pounds; I demand what each cost.*

The *House* cost $12x$, *Park* $5x$, *Garden* $1x$ } by the *Quest.*
Then $24x + 15x + 4x = 43x$
And $43x - 10000 = 5000 - 17x$

Theref. $60x = 15000$, by *Transposition*,

And $x = \frac{15000}{60} = 250$, by *Division*,

Theref. the *House* cost 3000 *l*, *Park* 1250 *l.* *Garden* 250 *l.*

NOTE.

That some of the preceding Questions with innumerable more such may be resolv'd much easier, by other Methods *Peculiar* to each;

But our design is only to acquaint the Reader with *General ways* of resolving such *Questions*, whence he may at Pleasure draw variety of *Particular* ones for his Practice.

And

And the Method of expressing each unknown *Quantity* by an unknown *Letter*, we take to be the most *General*; because it serves *universally* in all Cases, and does not in the least appear as if *contriv'd*: and therefore is preferable to any other.

Besides, this way cannot possibly strain the Imagination, nor mislead the Fancy, in *finding* other particular ways of *Notation*: For all the *Reasoning*, it supposes, is no more than what is immediately founded on the *Conditions of the Question*; and all the Deductions are Naturally drawn, only by reducing the *Equation* thus form'd, and substituting in the room of certain *unknown Quantities*, their Equivalents in different Expressions.

Those that are able to judge what Method of proceeding is the most Natural will see that we introduce only that which ought to be known, and deserves attention; the which we have insisted on the more that the Beginner might have clear and distinct Ideas of, and be throughly acquainted with the *Principal* Foundation of all *Analysis*, that he might hereafter the easier know how to draw useful Inferences, and make Methodical Applications in the more Abstruse Parts.

For *Analysis* of all other *Methods* is the most conducive to, if not the only means for the Discovery of *unknown Truths*; and not only the *Mathematics*, but in General all other *Sciences* would have bin imperfect without it; and the more they observe its *Laws*, the nearer they arrive to Perfection.

Of

Of Indetermined Questions.

Quest. 7. *A Composition is to be made of three Ingredients, the whole being 10 Pound Weight, the particulars are worth* 2 s. 4 s. *and* 8 s. *the Pound; 'tis required to mix them so, as that the Pound may be afforded for* 5 s. *How much of each must be taken?* Suppose x, y, z.

Then $x + y + z = 10$ }
And $2x + 4y + 8z = 50$ } by the *Conditions* of the *Quest.*

Here 'tis plain, the *Question* is *Indetermined*; because there are three *unknown Quantities*, and but two *Equations*.

Therefore let the Limits of any one of the unknown Quantities, suppose z, be *determined*, which may be done after this manner.

Since $x = 10 - y - z = \frac{50 - 4y - 8z}{2}$ by *Transp.*

And $y = 15 - 3z$, Therefore $z < 5$.

Also $x\ (= 10 - \overline{15 - 3z} - z) = 2z - 5$ ∴ $z > 2\frac{1}{2}$.

Whence these *Answers* in *whole Numbers*.

z,	x,	y,
3,	1,	6,
4,	3,	3.

Quest. 8. *A Refiner has 3 Ingots of Silver of different fineness, viz. of* 4, 6, *and of* 10 *Ounces fine, of which he would mix* 20 *pound weight, so as to make it* 8 *Ounces fine; how much must he take of each sort?* Suppose x, y, and z.

$$\left.\begin{array}{ll}\text{Then} & x+y+z=20\\ \text{And} & 4x+6y+10z=160\end{array}\right\}\text{ by the } \textit{Question};$$

$$\text{Theref. } x=20-y-z=\frac{160-6y-10z}{4}.$$

And $y=40-3z$, therefore $z<14$.

Also $x\ (=20-\overline{40-3z}-z)=2z-20 \therefore z>10$.

Whence these *Answers* in *whole Numbers.*

z	y	x
11	7	2
12	4	4
13	1	6

Quest. 9. *A Grocer having* 4 *sorts of Sugar, at* 12 d. 8 d. 6 d. *and* 4 d. *a Pound, desires to have a mixture of a* 100 *weight made out of them, so as that it may be afforded at* 10 d. *a Pound: how many Pounds must be taken of each?* Suppose x, y, z, u.

$$\left.\begin{array}{ll}\text{Then} & x+y+z+u=100\\ \text{And} & 12x+8y+6z+4u=10\times100\end{array}\right\}\text{ by the } \textit{Quest.}$$

$$\left.\begin{array}{ll}\text{Theref.} & y+z+u=100-x\\ \text{And} & 8y+6z+4u=1000-12x\end{array}\right\}\text{ by } \textit{Transp.}$$

$$\left.\begin{array}{ll}\text{Then} & 4y+4z+4u=400-4x\\ \text{And} & 8y+8z+8u=800-8x\end{array}\right\}\text{ from x by}\left\{\begin{array}{l}<\\>\end{array}\right\}\text{Coeff.}$$

$$\text{Then}\left\{\begin{array}{l}400-4x<1000-12x \therefore x<75\\ 800-8x>1000-12x \therefore x>50\end{array}\right.$$

Therefore supposing x (for instance) $=60$; let it be substituted in the *Equations* which express the *Conditions of the Question.*

Then

Then $60 + y + z + u = 100$
And $720 + 8y + 6z + 4u = 1000$

Theref. $y = 40 - z - u = \frac{280 - 6z - 4u}{8}$

And $z = 20 - 2u$, therefore $u < 10$

But $y\ (= 40 - \overline{20 - 2u} - u) = 20 + u$,

Therefore u has no other Limits, but may be any Number under 10.

Hence, in taking $x = 60$, we have these *Answers* in *whole Numbers.*

x	60	60	60	60	60	60	60	60	60
y	21	22	23	24	25	26	27	28	29
z	18	16	14	12	10	8	6	4	2
u	1	2	3	4	5	6	7	8	9

And by taking $x =$ to some other Number between its respective Limits, more *Answers* in *whole Numbers* may be found.

SCHOLIUM.

From *this* and the *foregoing* Chapters the several *Practical Rules* in *Common Arithmetic* are naturally deduced.

I. *The* Single Rule of Three, Direct *and* Inverse.

Wherein there are three *Terms* given, *two* of which are of the *same Denomination.*

And in *Questions* falling under this *Rule,* the *Terms* may be disposed in this *order,* viz.

$$\left.\begin{array}{l}\text{The } \textit{Querying Term}\\ \text{That of the } \textit{same kind,}\\ \text{And that of } \textit{different kind}\end{array}\right\}\text{ in the }\left\{\begin{array}{l}3\,d.\\ 1\,st.\\ 2\,d.\end{array}\right\}\textit{Place}\text{ : Then,}$$

$$\text{If the } 3d.\text{ be}\left\{\begin{array}{l}\textit{greater}\\ \textit{less}\end{array}\right\}\text{ than the } 1\,st.\text{ and requires that the } 4th.\text{ should be}\left\{\begin{array}{l}\textit{greater}\\ \textit{less}\end{array}\right\}\text{ than the } 2\,d.$$

$$\text{That is, if }\left\{\begin{array}{l}\textit{more}\\ \textit{less}\end{array}\right\}\text{ requires }\left\{\begin{array}{l}\textit{more}\\ \textit{less}\end{array}\right\}$$

The *Proportion* is *Direct*; But

$$\text{If the } 3d.\text{ be}\left\{\begin{array}{l}\textit{greater}\\ \textit{less}\end{array}\right\}\text{ than the } 1\,st.\text{ and requires that the } 4th.\text{ should be}\left\{\begin{array}{l}\textit{less}\\ \textit{greater}\end{array}\right\}\text{ than the } 2\,d.$$

$$\text{That is, if }\left\{\begin{array}{l}\textit{more}\\ \textit{less}\end{array}\right\}\text{ requires }\left\{\begin{array}{l}\textit{less}\\ \textit{more}\end{array}\right\}$$

The *Proportion* is *Inverse*: And

$$\text{The } \textit{Required Term} \text{ is} = \left\{\begin{array}{l}\dfrac{2\,d. \times 3\,d.}{1\,st.}\text{ in the } \textit{Direct}\\[2ex] \dfrac{1\,st. \times 2\,d.}{3\,d.}\text{ in the } \textit{Inverse}\end{array}\right\}\textit{Rule.}$$

But in the *Inverse Rule*, if the *Terms* be placed thus, *viz.*

$$\left.\begin{array}{l}\text{The } \textit{Querying Term}\\ \text{That of the } \textit{same kind}\\ \text{That of } \textit{different kind}\end{array}\right\}\text{ in the }\left\{\begin{array}{l}1\,st.\\ 3\,d.\\ 2\,d.\end{array}\right\}\textit{Place}\text{ ;}$$

Then the Work is as in the *Direct Rule*.

COROLLARY.

When the *Rule of Three Direct* has 1 for the 1st. *Term*, 'tis usually called the *Rule of Practice*, from its frequent use and ready Performances in Common Affairs; for,

1. If

1. If the *Price*, &c. of the *Integer* be an *Aliquot part* of a *Pound*, *Shilling*, &c. as,

If 1 *yd.* : 6 *d.* :: 420 *yd.* : *w*? But $6\,d. = \frac{6}{12}\,s = \frac{1}{2}\,s.$

$\therefore w \left(= \frac{\frac{1}{2}s \times 420}{1\,s.} = \frac{s}{2} \times \frac{420}{1} = \frac{420}{2}s = 210\,s\right) = 10\,l.\ 10\,s.$

If 1 *yd.* : 4 *d.* :: 63 *yd.* : *w*? But $4\,d. = \frac{4}{12}\,s = \frac{1}{3}\,s.$

$\therefore w \left(= \frac{\frac{1}{3}s \times 63}{1\,s.} = \frac{s}{3} \times \frac{63}{1} = \frac{63}{3}s\right) = 21\,s = 1\,l.\ 1\,s.$

If 1 *yd.* : 10 *s.* :: 34 *yd.* : *w*? but $10\,s = \frac{10}{20}\,l. = \frac{1}{2}\,l.$

$\therefore w \left(= \frac{10\,s \times 34}{20\,s} = \frac{\frac{1}{2}l \times 34}{1\,l.} = \frac{l}{2} \times \frac{34}{1} = \frac{34}{2}l\right) = 17\,l.$

If 1 *yd.* : 2 *s* :: 56 *yd.* : *w*? but $2\,s = \frac{2}{20}\,l. = \frac{1}{10}\,l.$

$\therefore w \left(= \frac{1}{10}l. \times \frac{56}{1}l. = \frac{56}{10}\right) = 5\,l. + \overline{6 \times 2\,s} = 5\,l.\ 12\,s.$

2. If the *Price*, &c. of the *Integer* is Composed of the *Aliquot Parts* of a *Pound*, *Shilling*, &c. as,

If 1 *yd* : 9 *d* :: 84 *yd* : *w*? but $9\,d. \left(= \frac{6}{12}s + \frac{3}{12}s\right) = \frac{1}{2}s + \frac{1}{4}s$

$\therefore w \left(= \frac{84}{2}s + \frac{84}{4}s = 42\,s + 21\,s = 63\,s\right) = 3\,l.\ 3\,s.$

If 1 *yd* : 7 *s* :: 264 *yd* : *w*? but $7\,s. \left(= \frac{5}{20}l. + \frac{2}{20}l. =\right) \frac{1}{4}l. + \frac{1}{10}l.$

$\therefore w \left(= \frac{264}{4}l + \frac{264}{10}l = 66\,l + 26\,l + \overline{4\,s \times 2\,s}\right) = 92\,l.\ 8\,s.$

3. Also

3, Alſo if 1 lb. : *n* f. :: 1 C. (= 112 lb.) *w* ?
Since 112 f. (= 2 *s* + 4 *d*) = 2 *s* + 1 *groat*,
Therefore $w = n \times \overline{2s + 1g}$ ſought,

The *manner* of *Operation* in the other *Caſes* of the *Rule of Practice* cannot but appear evident to them that throughly underſtand what is here already delivered, therefore this alone was thought ſufficient; eſpecially, ſince that Accurate *Penman* and moſt Ingenious *Accomptant* Mr. *George Shelley*, in his late *Supplement* to Mr. *Wingate*'s *Arithmetic*, has (amongſt variety of very uſeful *Compendious Rules* for the ready Solving of ſuch *Queſtions*, as daily occur in Common and Ordinary Affairs) particularly inſiſted on the ſeveral parts of the *Rule of Practice*, the which he that would be farther inform'd would do well to conſult.

II. *The* Compound Rule of Three, Direct *and* Inverſe.

Let the *Terms* of the *Suppoſition* with their correſponding ones of the *Demand*, be ſo placed, as to appear directly Stated; then 'twill be either;

1			2			3		
A,	B,	C,	A,	B,	C,	A,	B,	C,
a,	*b*,	,	*a*,	,	*c*,	,	*b*,	*c*,

where $c = \frac{C\,a\,b}{A\,B}$; $b = \frac{c\,A\,B}{C\,a}$; $a = \frac{c\,A\,B}{C\,b}$;

For

For

$$1.\ \left\{\begin{array}{l} A : C :: a : w \\ B : w :: b : c \end{array}\right\} \therefore AB : C :: ab : c = \frac{C\,ab}{AB}.$$

$$2.\ \left\{\begin{array}{l} C : A :: c : w \\ a : w :: B : b \end{array}\right\} \therefore Ca : A :: cB : b = \frac{cAB}{Ca}.$$

$$3.\ \left\{\begin{array}{l} C : A :: c : w \\ b : w :: B : a \end{array}\right\} \therefore Cb : A :: cB : a = \frac{cAB}{Cb}.$$

EXAMPLE 1.

If the / The $\left\}\right.$ *Men* $\left\{\begin{array}{l} M \\ m \end{array}\right\}$ in the *Time* $\left\{\begin{array}{l} T \\ t \end{array}\right\}$ spend *Pounds* $\left\{\begin{array}{l} P ; \\ p ? \end{array}\right.$

Then $MT : P :: mt : p = \frac{Pmt}{MT}$

For $M : m :: P : \frac{Pm}{M}$.

And $T : t :: \frac{Pm}{M} : \frac{Pmt}{MT} = p$ sought.

EXAMPLE 2.

If a / The $\left\}\right.$ *Body* $\left\{\begin{array}{l} L \\ l \end{array}\right\}$ *Long*, $\left\{\begin{array}{l} B \\ b \end{array}\right\}$ *Broad*, $\left\{\begin{array}{l} T \\ t \end{array}\right\}$ *Thick*, / *Weighs* $\left\{\begin{array}{l} W ; \\ w ? \end{array}\right.$

Then $LBT : W :: lbt : \frac{Wlbt}{LBT} = w$.

For $L : l :: W : \frac{Wl}{L}$.

Also $B : b :: \frac{Wl}{L} : \frac{Wlb}{LB}$.

And $T : t :: \frac{Wlb}{LB} : \frac{Wlbt}{LBT} = w$ sought.

III. *The*

III. *The* Rule *of* Fellowship.

Example in the *Single Rule of Fellowship.*

$$\text{Suppose}\left\{\begin{matrix}\mathrm{M}\\ M\\ m\\ \&c.\end{matrix}\right\}\text{ put }\left\{\begin{matrix}\mathrm{L}\\ L\\ l\\ \&c.\end{matrix}\right\}\text{ into the } \textit{Joint Stock};$$

Let x be the whole *Gain*, or *Loss*:

Required the *Gain*, or *Loss* of each of them. Then,

$$\mathrm{L}+L+l,\&c. : x :: \left\{\begin{matrix} \mathrm{L} : \dfrac{\mathrm{L}x}{\mathrm{L}+L+l,\&c.} = \mathrm{M}\text{'s} \\ L : \dfrac{Lx}{\mathrm{L}+L+l,\&c.} = M\text{'s} \\ l : \dfrac{lx}{\mathrm{L}+L+l,\&c.} = m\text{'s} \\ \&c. \end{matrix}\right\} \textit{Gain, or Loss.}$$

$$\text{For } \mathrm{L}+L+l,\&c. : x :: \mathrm{L}+L+l,\&c. : \frac{\mathrm{L}x+Lx+lx,\&c.}{\mathrm{L}+L+l,\&c.} = x.$$

Example in the *Compound Rule of Fellowship.*

$$\text{Sup.}\left\{\begin{matrix}\mathrm{M}\\ M\\ m\\ \&c.\end{matrix}\right\}\text{ put in }\left\{\begin{matrix}\mathrm{L}\\ L\\ l\\ \&c.\end{matrix}\right\}\text{ for the } \textit{Time}\left\{\begin{matrix}\mathrm{T}\\ T\\ t\\ \&c.\end{matrix}\right\};\text{ Then,}$$

$$\mathrm{LT}+LT+lt,\&c. : x :: \left\{\begin{matrix} \mathrm{LT} : \dfrac{\mathrm{LT}x}{\mathrm{LT}+LT+lt,\&c.} = \mathrm{M}\text{'s} \\ LT : \dfrac{LTx}{\mathrm{LT}+LT+lt,\&c.} = M\text{'s} \\ lt : \dfrac{ltx}{\mathrm{LT}+LT+lt,\&c.} = m\text{'s} \\ \&c. \end{matrix}\right\} \textit{Gain, or Loss.}$$

$$\text{For } \mathrm{LT}+LT,\&c. : x :: \mathrm{LT}+LT,\&c. : \frac{\mathrm{LT}x+LTx}{\mathrm{LT}+LT,\&c.} = x.$$

IV. *Of* Reduction *and* Exchanges.

Ex. 1. If 100 Ells of *Antwerp* = 75 Yards of *London*; How many Yards of *London* = 27 Ells of *Antwerp*?

$$100A = 75L, \therefore 1A = \frac{75}{100}L, \text{and } 27A = \frac{27\times75}{100}L = 20\tfrac{1}{4}.$$

Ex. 2. If

If	35	Ells of	*Vienna* = 24	Ells of	*Lyons*
	3		*Lyons* = 5		*Antwerp*
	100		*Antwerp* = 125		*Frankfort*
How many —			*Vienna* = 50		*Frankfort*

$$35V = 24L \therefore 1L = \frac{35}{24}V$$

$$3L = 5A \therefore 1L = \frac{5}{3}A = \frac{35}{24}V \therefore 1A = \frac{3\times35}{5\times24}V.$$

$$100A = 125F \therefore 1A = \frac{125}{100}F = \frac{3\times35}{5\times24}V, \text{ Therefore,}$$

$$1F = \frac{100\times3\times35}{125\times5\times24}V \therefore 50F = \frac{50\times100\times3\times35}{125\times5\times24}V = 35.$$

Ex. 3. If

Ex. 3. If	6 lb. of *Pepper*	is worth	12 lb. of *Ginger*
	24 lb. of *Ginger*		4 lb. of *Cinnamon*
	5 lb. of *Cinnamon*		60 lb. of *Sugar*
	1 lb. of *Sugar*		6 *Pence Sterling.*

How many —— *Pence* is —— 112 lb. of *Pepper* worth?

Since,

$$\left.\begin{array}{l} 6P = 12G \therefore 1G = \frac{6}{12}P \\ 24G = 4C \therefore 1G = \frac{4}{24}C \end{array}\right\} \therefore 1C = \frac{24\times6}{12\times4}P.$$

$$5C = 60S \therefore 1C = \frac{60}{5}S = \frac{24\times6}{12\times4}P \therefore 1S = \frac{24\times6\times5}{12\times4\times60}P.$$

$$1S = 6d = \frac{24\times6\times5}{12\times4\times60}P \therefore 1P = \frac{12\times4\times60\times6}{24\times6\times5}d.$$

$$\text{And } 112P = \frac{12\times4\times60\times6\times112}{24\times6\times5}d = 2688\,d.$$

That is, 112lb. of *Pepper* is worth 11l. 4s. *Sterling*.

V. *The Rule of* Alligation.

This we ſhall wholly omit as Imperfect, becauſe the *Queſtions* falling under it are *Indetermined ones*, which this *Rule*, as commonly delivered, cannot *fully Solve*; tho' it may perhaps give one or more *True Anſwers*, yet they may not be thoſe that are required for preſent occaſion; of which ſeveral Inſtances might eaſily be given, as the Learned Dr. *Wallis* has done in *Ch.* 58. *Vol.* 2. of his *Works*.

But as to the *Solutions* of ſuch *Queſtions*, we refer the Reader to the Method already given for Solving *Indetermined Queſtions*, of which the *Rule of Alligation* is but an *Example*.

VI. *The* Rule *of* Falſe Poſition.

1. *Single Poſition*.

Let A repreſent the *Abſolute Number* given;
P any *Suppoſed Number*;

S, The

S { The *Sum* produced by P, when order'd according to the *Conditions of the Question* :

Then $\frac{PA}{S}$ is the *Number sought.*

This is evident from the *Operation* it self.

2. *Double Position.*

Let P, *p*, be the *two Positions* ;

E, *e*, { Their respective *Errors*, when order'd according to the *Question* ;

n The *Number sought* : Then,

If $\pm E, \pm e, n = \frac{Pe \backsim pE}{E \backsim e}$; If $\pm E, \mp e, n = \frac{Pe + pE}{E + e}$.

For suppose it were required to know what *Quantity* into *a* would give *a n*.

1. If $P = n - x$ ——————— $p = n - z$

Then $na - xa$ is not $= na$ ——— $na - za$ is not $= na$.

Therefore, $E = na - \overline{na - xa}$ ——— $e = na - \overline{na - za}$.

And by *Subduction*, because the *Signs* are *alike*,

$$xa \backsim za = E \backsim e$$

But $Pe = nza - xza$, and $pE = nxa - zxa$,

Therefore, by *Subduction also*,

$Pe \backsim pE \; (= nza \backsim nxa) = n \times \overline{E \backsim e}$

Consequently $n = \frac{Pe \backsim pE}{E \backsim e}$, by *Division*.

2. If $P = n + x$ ——————— $p = n - z$

Then $na + xa$ is not $= na$, $na - za$ is not $= na$

Theref. $E = \overline{na + xa} - na$, ——— $e = na - \overline{na - za}$

And by *Addition*, because the *Signs* are *unlike*,

$$xa + za = E + e$$

But $Pe = z a n + z a x$, and $pE = x a n - z a x$

Therefore by *Addition* also,

$Pe + pE \; (= x a n + z a n) = \overline{E + e} \times n.$

Consequently $n = \dfrac{Pe + pE}{E + e}$ by *Division*.

II. OF QUADRATIC EQUATIONS.

1. If $xx - ax \left\{ \begin{matrix} - \\ + \end{matrix} \right\} b = 0$, or $xx = ax \left\{ \begin{matrix} + \\ - \end{matrix} \right\} b$; put $y + \frac{1}{2}a = x$.

Then $yy + ay + \frac{1}{4}aa = ay + \frac{1}{2}aa \left\{ \begin{matrix} + \\ - \end{matrix} \right\} b.$

Theref. $yy = \frac{1}{4}aa \left\{ \begin{matrix} + \\ - \end{matrix} \right\} b$, and $y = \overline{\frac{1}{4}aa \left\{ \begin{matrix} + \\ - \end{matrix} \right\} b}\Big|^{\frac{1}{2}}$

Conseq. $x = + \frac{1}{2}a \pm \overline{\frac{1}{4}aa \left\{ \begin{matrix} + \\ - \end{matrix} \right\} b}\Big|^{\frac{1}{2}}$

2. If $xx + ax \left\{ \begin{matrix} - \\ + \end{matrix} \right\} b = 0$, or $xx = -ax \left\{ \begin{matrix} + \\ - \end{matrix} \right\} b$; put $y - \frac{1}{2}a = x$.

Then $yy - ay + \frac{1}{4}aa = -ay + \frac{1}{2}aa \left\{ \begin{matrix} + \\ - \end{matrix} \right\} b$

Theref. $yy = \frac{1}{4}aa \left\{ \begin{matrix} + \\ - \end{matrix} \right\} b$, and $y = \overline{\frac{1}{4}aa \left\{ \begin{matrix} + \\ - \end{matrix} \right\} b}\Big|^{\frac{1}{2}}$

Conseq. $x = -\frac{1}{2}a \pm \overline{\frac{1}{4}aa \left\{ \begin{matrix} + \\ - \end{matrix} \right\} b}\Big|^{\frac{1}{2}}.$

Therefore if V be put for the *Sign* of any *Term*, and Λ for the contrary, all *Forms of Quadratics*, with their *Solutions*, will be reduc'd to this one.

If $xx \, V \, ax \, V b = 0$, then $x = \Lambda \frac{1}{2}a \pm \overline{\frac{1}{4}aa \, \Lambda \, b}\Big|^{\frac{1}{2}}$

These *Solutions* may also be found thus;

Form $\left\{ \begin{matrix} 1. \; rr \mp sr = -\alpha \\ 2. \; rr \mp dr = +\alpha. \end{matrix} \right.$

Since

Since $\frac{1}{2}s \pm \frac{1}{2}d = \left\{\begin{matrix} a \\ e \end{matrix}\right\}$, Therefore,

$\overline{\frac{1}{2}s + \frac{1}{2}d} \times \overline{\frac{1}{2}s - \frac{1}{2}d} = \overline{\frac{1}{4}s^2 - \frac{1}{4}d^2} = æ,$

And $4æ = s^2 - d^2$, Therefore,

$d = \overline{s^2 - 4æ}|^{\frac{1}{2}}$, and $s = \overline{d^2 + 4æ}|^{\frac{1}{2}}$

But $\overline{\frac{1}{4}d^2}|^{\frac{1}{2}}$ $(= \frac{1}{2}d) = \overline{\frac{1}{4}s^2 - æ}|^{\frac{1}{2}}$, and

$\overline{\frac{1}{4}s^2}|^{\frac{1}{2}}$ $(= \frac{1}{2}s) = \overline{\frac{1}{4}d^2 + æ}|^{\frac{1}{2}}$.

Theref. 1. $+a, +e = +\frac{1}{2}s \pm \overline{\frac{1}{4}ss - æ}|^{\frac{1}{2}}$

$-a, -e = -\frac{1}{2}s \mp \overline{\frac{1}{4}ss - æ}|^{\frac{1}{2}}$

2. $+a, -e = +\frac{1}{2}d \pm \overline{\frac{1}{4}dd + æ}|^{\frac{1}{2}}$

$-a, +e = -\frac{1}{2}d \mp \overline{\frac{1}{4}dd + æ}|^{\frac{1}{2}}$

EXAMPLES.

1. If $xx - 4x = 12$; then $x = +6$, or -2.
2. If $xx - 2x = 7$; then $x = 1 \pm 8^{\frac{1}{2}}$, or $1 \pm 2 \times 2^{\frac{1}{2}}$
3. If $x^2 - ax + x = b - c - d$; Then,

$x = \frac{1}{2}a - \frac{1}{2} \pm \overline{\frac{1}{4}a^2 - \frac{1}{2}a + \frac{1}{4} + b - c - d}|^{\frac{1}{2}}$

4. If $x^2 + \frac{a}{b}x = c$, then;

$x = \pm \overline{\frac{aa}{4bb} + c}|^{\frac{1}{2}} - \frac{a}{2b} = \frac{\pm \overline{a^2 + 4b^2c}|^{\frac{1}{2}} - a}{2b}$

5. $x^2 - abx - cdx + efx - gbx = lmn + pqr - stw$

Then $x = \frac{1}{2}z \pm \overline{\frac{1}{4}zz + lmn + pqr - stw}|^{\frac{1}{2}}$

6. $x^4 - ax^2 = b$, let $x^2 = y$, then $y^2 - ay = b$, and

$y = \frac{1}{2}a \pm \overline{\frac{1}{4}a^2 + b}|^{\frac{1}{2}}$ ∴ $x = \overline{\frac{1}{2}a \pm \overline{\frac{1}{4}aa + b}|^{\frac{1}{2}}}|^{\frac{1}{2}}$.

7. x^6 —

7. $x^6 - ax^3 = b$, let $x^3 = y$, then $y^2 - ay = b$, and $y = \frac{1}{2}a \pm \overline{\frac{1}{4}aa + b}|^{\frac{1}{2}} \therefore x = \overline{\frac{1}{2}a \pm \overline{\frac{1}{4}aa + b}|^{\frac{1}{2}}}|^{\frac{1}{3}}$.

8. $nx^2 - ax = b$, then $x = \frac{\frac{1}{2}a \pm \overline{\frac{1}{4}aa + nb}|^{\frac{1}{2}}}{n}$

For $x^2 = \frac{ax}{n} + \frac{b}{n}$, $\therefore x = \frac{\frac{1}{2}a}{n} \pm \overline{\frac{\frac{1}{4}aa}{nn} + \frac{b}{n}}|^{\frac{1}{2}}$

That is, $nx = \frac{1}{2}a \pm \overline{\frac{1}{4}a^2 + nb}|^{\frac{1}{2}}$, theref. $x =$ &c.

9. $nx^2 \pm 2abx = cdn$, then $x = \frac{\overline{\pm a^2b^2 + n^2cd}|^{\frac{1}{2}} - ab}{n}$

10. $n^2x^2 - m^2x^2 - a^3x = -c^4$, then

$x = \frac{\frac{1}{2}a^3 \pm \overline{\frac{1}{4}a^6 - n^2c^4 + m^2c^4}|^{\frac{1}{2}}}{nn - mm}$.

11. $5xx - 4x = 3$, then $x = \frac{2 \pm \overline{19}|^{\frac{1}{2}}}{5}$

12. $nnx^4 + dx^2 - ex^2 = + a^6 - b^6 + c^6$, let $d - e = z$.

Then $x = \overline{\frac{-\frac{1}{2}z \pm \overline{\frac{1}{4}zz + a^6 - b^6 + c^6 \times nn}|^{\frac{1}{2}}}{nn}}|^{\frac{1}{2}}$

Quest. 1. *A Man bought a Horse, which he sold again for* 24 l. *and found he had gain'd as much* per Cent. *as the Horse cost him; I demand what the Horse cost at first.* Suppose it cost y Pounds.

Let $24 = a$, $100 = b$,

Then $y : a - y :: b : \frac{ba - by}{y} = y$, by the *Quest.*

Therefore

Therefore $yy + by = ba$, by *Mult.* and *Tranfp.*

Confeq. $y = \overline{ba + \frac{1}{4}bb}|^{\frac{1}{2}} - \frac{1}{2}b$.

Or, $y = \overline{2400 + 2500}|^{\frac{1}{2}} - 50 = 20\,l.$ fought.

Queft. 2. *A Vintner bought a parcel of Clarret and Rhenifh Wine, which together coft him* 120 l. *as for the feparate prices, 'tis only known that their Product was equal to* 2700 l. *Required what each coft?*

The *Clarret* coft c
Then *Rhenifh* coft $120 - c$ } by the *Queftion.*
And $\overline{120c - cc = 2700}$

Theref. $c = \overline{\frac{120 \times 120}{4} - 2700}|^{\frac{1}{2}} + \frac{120}{2} = 90$ fought.

That is, The *Clarret* coft 90 *l.* *Rhenifh* 30 *l.*

Queft. 3. *Having bought a Quantity of Goods, viz.* Pepper *Cloves, Ambergreice, their prices were forgot, but 'tis known that they were continually Proportional, and that the Pepper and Cloves coft* 10 l. *the Ambergreice* 24 l. *more than the Cloves; 'tis required what each coft.*

The *Pepper* coft p, *then Cloves coft* $10 - p$.

Then $p : 10 - p :: 10 - p : \frac{100 - 20p + pp}{p}$ } by the *Queft.*
And $\left(\frac{100 - 20p + pp}{p} - \overline{10 - p} =\right) \frac{100 - 30p + 2pp}{p} = 24$

Therefore $pp - 27p = -50$, by *Mult. Tranfp.* and *Divif.*

And $p = \frac{27}{2} - \overline{\frac{729 - 200}{4}}|^{\frac{1}{2}} = 2$.

Therefore *Pepper* coft 2 *l.* *Cloves* 8 *l.* *Ambergreice* 32 *l.*

III. OF

III. OF CUBIC EQUATIONS.

All *Cubic Equations*, whose *second Term* is destroyed, may be reduced into these *Forms* $xxx \pm ax - b = 0$, That is, $xxx \pm ax = +b$.

Because in *those* where the *Absolute known Quantity* is *Negative*, there needs no more than making the *Roots* which were *Affirmative* in those, *Negative* in these.

If $x^3 \pm ax - b = 0$, then $x^2 \pm a - \frac{b}{x} = 0$.

And since the $\left\{\begin{matrix}\text{Diff.}\\ \text{Sum}\end{matrix}\right\}$ *of any two like Cubes, is exactly divisible by the* $\left\{\begin{matrix}\text{Diff.}\\ \text{Sum}\end{matrix}\right\}$ *of their Roots*; Therefore,

Suppose, $b = m^3 \mp n^3$, and $x = m \mp n$, then the *Equation* will be $\mp 3nm \pm a = 0$, theref. $n = \frac{a}{3m}$

And $m^3 \mp (n^3 =) \frac{a^3}{27m^3} = b$, then $m^6 - bm^3 = \pm \frac{1}{27}a^3$

Theref. $m^3 = \frac{1}{2}b + \sqrt[2]{\frac{1}{4}bb \pm \frac{1}{27}a^3}$, but $m^3 \pm n^3 = b$,

Then $b = \frac{1}{2}b + \sqrt[2]{\frac{1}{4}bb \pm \frac{1}{27}a^3} \mp n^3$, Or

$n^3 = \pm \frac{1}{2}b \pm \sqrt[2]{\frac{1}{4}bb \pm \frac{1}{27}a^3}$, and $m \mp n$, *i. e.*

$$x = \sqrt[3]{\frac{b}{2} + \sqrt[2]{\frac{bb}{4} \pm \frac{a^3}{27}}} \mp \sqrt[3]{\mp \frac{b}{2} \pm \sqrt[2]{\frac{bb}{4} \pm \frac{a^3}{27}}}$$

Therefore,

1. If $xxx + ax \pm b = 0$, Then,

$$x = \mp \sqrt[3]{\frac{b}{2} + \sqrt[2]{\frac{bb}{4} + \frac{a^3}{27}}} \pm \sqrt[3]{-\frac{b}{2} + \sqrt[2]{\frac{bb}{4} + \frac{a^3}{27}}}$$

2. If

2. If $xxx - ax \pm b = 0$, Then

$$x = \mp\sqrt[3]{\frac{b}{2} + \sqrt[2]{\frac{bb}{4} - \frac{a^3}{27}}} \mp \sqrt[3]{\frac{b}{2}b - \sqrt[2]{\frac{b^2}{4} - \frac{a^3}{27}}}$$

Having discovered one of the Roots of an Equation, *the rest may be found by* Division.

And as for *Equations* of *higher Dimensions*, we shall omit the like Methods that might be given for their *Solutions*: because more intricate and tedious for Practice, than the Rules of *Numeral Exegesis* delivered by *Vieta*, *Harriot*, and *Oughtred*; which are yet abundantly improved by the Method of *Infinitly Converging Series*, (an *Universal* way of *Extracting the Roots* of any *Equation* whatsoever.) And of this Method we shall in its proper Place, give *General Examples*, whence all possible *Particular* ones may be drawn.

NOTE.

That *Equations* of *high Dimensions* may oftentimes be advantageously reduced *lower*, by some of the former *Rules*, or those that may easily be deduced from them; according to the Methods of *Hudden*, *Merry*, and others; and then the *Solution* will be less troublesome.

CHAP. VIII.

Of Arithmetic Progression.

DEFINITION I.

A Continued Arithmetic Proportion, *that is, where the Terms do* Increase *and* Decrease *by equal* Differences, *is call'd* Arithmetic Progression.

Thus $\left\{\begin{matrix} a,\ a+d,\ a+2d,\ a+3d, \text{\&c. } \textit{Increasing} \\ a,\ a-d,\ a-2d,\ a-3d, \text{\&c. } \textit{Decreasing} \end{matrix}\right\}$ by d.

SCHOLIUM 1.

But since this *Progression* is only a Compound of two *Series*,

viz. of $\left\{\begin{matrix} \textit{Equals} & a, & a, & a, & a, & a, \\ \textit{Arith. Prop.} & 0, & \pm d, & \pm 2d, & \pm 3d, & \pm 4d, \end{matrix}\right\}$ &c.

Therefore the most Natural *Arithmetic Progression* is that which begins with 0;

As $0, \pm d, \pm 2d, \pm 3d, \pm 4d, \pm 5d,$ $\left\{\begin{matrix} \textit{Increasing} \\ \textit{Decreasing.} \end{matrix}\right.$

SCHOLIUM 2.

In an *Arithmetic Progression*;

If $\left\{\begin{matrix} a \\ d \\ n \\ l \\ s \end{matrix}\right\}$ be the $\left\{\begin{matrix} \textit{First Term} \\ \textit{Common Difference} \\ \textit{Number of Terms} \\ \textit{Last Term} \\ \textit{Sum of all the Terms} \end{matrix}\right.$

Then

Then any *Three* of these *Terms* being *given*, the other *Two* are easily *found*.

And the several *Cases* are reducible into *Ten Propositions*, which are all Solved by the *Two* following *Lemmata*.

LEMMA I.

In any *Arithmetic Progression*;

'Tis $1 : \frac{n}{2} :: a+l : s$.

$$\text{For } \left\{\begin{matrix} a \\ a+d \\ a+2d \\ a+3d \\ \&c. \end{matrix}\right\} + \left\{\begin{matrix} l \\ l-d \\ l-2d \\ l-3d \\ \&c. \end{matrix}\right\} = \left\{\begin{matrix} a+l \\ a+l \\ a+l \\ a+l \\ \&c. \end{matrix}\right\}$$

Therefore $s + s = \overline{a+l} \times n$

That is, $2s = \overline{a+l} \times n$.

Consequently, $1 : \frac{n}{2} :: a+l : s$.

COROLLARIES.

1. $a = \frac{2s}{n} - l = \frac{2s - nl}{n} = \overline{2s - nl} \times \frac{1}{n}$

2. $n = \frac{2s}{a+l} = 2 \times \frac{s}{a+l} = 2s \times \frac{1}{a+l}$.

3. $l = \frac{2s}{n} - a = \frac{2s - na}{n} = \overline{2s - na} \times \frac{1}{n}$.

4. $s = \frac{n}{2} \times \overline{a+l} = \frac{n \times \overline{a+l}}{2} = \frac{na + nl}{2} = n \times \frac{a+l}{2}$.

LEMMA 2.

In any *Arithmetic Progression*;
'Tis $1 : n-1 :: d : l-a$.

For, $a, a+d, a+2d, a+3d, a+\overline{n-1}\times d=l$.
That is, $\overline{n-1}\times d=l-a$, by *Transpos.*
Therefore $1 : n-1 :: d : l-a$.

COROLLARIES.

1. $a=l-\overline{n-1}\times d=l-nd+d$.
2. $n=\frac{l-a}{d}+1=\frac{l-a+d}{d}$.
3. $d=\frac{l-a}{n-1}=\overline{l-a}\times\frac{1}{n-1}$.
4. $l=a+\overline{n-1}\times d=a+nd-d$.

PROPOSITION 1.

Given a, d, n; Required l, s.

SOLUTION.

1. $l=a+nd-d=\frac{2s-na}{n}$ by *Lem.* 2d. and 1*st*.
Then $na+nnd-nd=2s-na$, by *Mult.*
And $2s=2na+nnd-nd$, by *Transp.*
2. Theref. $s=na+\frac{nnd-nd}{2}$ by *Division*.

P R O.

PROPOSITION 2.

Given, a, d, l; Required, n, s.

SOLUTION.

1. $n = \frac{l - a + d}{d} = \frac{2s}{a + l}$ by *Lem.* 2d. and 1st.

Then $2ds = ll + ld - a^2 + ad$, by *Mult.*

2. Theref. $s = \frac{ll + ld - a^2 + ad}{2d}$, by *Division.*

PROPOSITION 3.

Given, a, d, s; Required n, l.

SOLUTION.

Since $l = \frac{2s - na}{n} = a + nd - d$, by *Lem.* 1. and 2.

Theref. $nnd + 2na - nd = 2s$ by *Mult.* and *Transp.*

And $nn + \frac{2a - d}{d} n = \frac{2s}{d}$ by *Division,*

1. Then $n = \frac{\overline{aa + \frac{1}{4}dd - ad + 2ds}|^{\frac{1}{2}} - a}{d} + \frac{1}{2}$.

And because $n = \frac{2s}{a + l} = \frac{l - a + d}{d}$ by *Lem* 1. and 2.

Theref. $ll + dl = 2ds - ad + aa$, by *Mult.* and *Transf.*

2. Then $l = \overline{2ds - ad + aa + \frac{1}{4}dd}|^{\frac{1}{2}} - \frac{1}{2}d$, by *Compl.* the *Square*, and *Evolut.*

PRO-

PROPOSITION 4.

Given, a, l, s; Required n, d.

SOLUTION.

1. $n = \frac{2s}{l+a} = \frac{l-a+d}{d}$ by *Lem.* 1 and 2.

Then $2ds - ld - ad = ll - aa$, by *Mult.* and *Transf.*

2. Theref. $d = \frac{ll - aa}{2s - l - a}$ by *Division.*

PROPOSITION 5.

Given a, n, s; Required l, d.

SOLUTION.

1. $l = \frac{2s - na}{n} = a + nd - d$ by *Lem.* 1. and 2.

Then $nnd - nd = 2s - 2na$ by *Mult.* and *Transf.*

2. Theref. $d = \frac{2s - 2na}{nn - n}$, by *Division.*

PROPOSITION 6.

Given a, n, l; Required d, s.

SOLUTION.

1. $d = \frac{l-a}{n-1} = \overline{l-a} \times \frac{1}{n-1}$, by *Lemma* 2.

2. $s = \frac{na + nl}{2} = \overline{a+l} \times \frac{n}{2}$, by *Lemma* 1.

PRO-

PROPOSITION 7.

Given d, l n; Required a, s.

SOLUTION.

1. $a = l - nd + d = \frac{2s - nl}{n}$ by *Lem.* 2. and 1.

Then $2s = 2nl - nnd + nd$, by *Mult.* and *Transf.*

2. Theref. $s = \frac{2nl - nnd + nd}{2}$ by *Division.*

PROPOSITION 8.

Given d, n, s; Required a, l.

SOLUTION.

Since $l = a + nd - d = \frac{2s - na}{n}$, by *Lem.* 2. and 1.

Then $2na = 2s - nnd + nd$, by *Mult.* and *Transp.*

1. Theref. $a = \frac{2s - nnd + nd}{2n}$, by *Division.*

And since $a = l - nd + d = \frac{2s - nl}{n}$, by *Lem.* 2. and 1.

Then $2nl = 2s + nnd - nd$, by *Mult.* and *Transp.*

2. Theref. $l = \frac{2s + nnd - nd}{2n}$, by *Division.*

PROPOSITION 9.

Given, d, l, s; Required a, n.

SOLUTION.

Since $n = \frac{2s}{a+l} = \frac{l-a+d}{d}$, by *Lem.* 1. and 2.

Then $aa - ad = ll + ld - 2ds$, by *Mult.* and *Transf.*

1. Theref. $a = \pm \overline{ll + ld - 2ds + \frac{1}{4}dd}|^{\frac{1}{2}} + \frac{1}{2}d$,

And because, $a = l - nd + d = \frac{2s - nl}{n}$, by *Lem.* 2. & 1.

Theref. $-nnd + 2nl + nd = 2s$, by *Mult.* & *Transf.*

And $-nn + \frac{2l+d}{d}n = \frac{2s}{d}$ by *Division.*

2. Then $n = \frac{1}{2} + \frac{l \mp \overline{ll + \frac{1}{4}dd + ld - 2ds}|^{\frac{1}{2}}}{d}$

PROPOSITION 10.

Given, n, l, s; Required, a, d.

SOLUTION.

1. $a = \frac{2s - nl}{n} = l - nd + d$ by *Lem.* 1. and 2.

Then $2nl - 2s = nnd - nd$, by *Mult.* and *Transp.*

2. Theref. $d = \frac{2nl - 2s}{nn - n}$ by *Division.*

PROB.

PROBLEM I.

To find the Sum of the Powers *of any* Arithmetic Progression.

PREPARATION.

Suppose n the *Index* of the *Power*.

Let each *Term* of the *Progression* be raised to each *Power*, under that whose *Sum* is sought.

And let the *Sum* of each *Rank* so rais'd be *multiplied* by the *Multiple* of the *like Dimension* of a in $\overline{a+d}|^{n+1}$

Put z for the *Sum* of all the *Products*.

And m for the *Multiple* of a^n in the *Power* $\overline{a+d}|^{n+1}$

SOLUTION.

Then $\dfrac{\overline{l+d}|^{n+1} - \overline{a^{n+1} + nd^{n+1}} + z}{m}$ is the *Sum* of any *Series* of *Powers* whose *Roots* are *Arithmetically Proportional*.

For Suppose the *Sum* of the *Cubes* of this *Arithmetic Progression*, a, $a+d$, $a+2d$, $a+3d$, was required.

1. $\overline{a+d}|^{n+1} = \overline{a+d}|^4 = a^4 + 4a^3d + 6a^2d^2 + 4ad^3 + d^4$

And the *Sum* of this *Series* is $4a + 6d$

Which *Multiply* by $4d^3$ (the *Multiple* of a in $\overline{a+d}|^{3+1}$)

The *Product* will be $16ad^3 + 24d^4$

Also the *Sum* of their *Squares* is $4a^2 + 12ad + 14d^2$

Which *Mult.* by $6d^2$ (the *Multiple* of a^2 in $\overline{a+d}|^{3+1}$)

The *Product* will be $24a^2d^2 + 72ad^3 + 84d^4$

Theref. $z = 24a^2d^2 + 88ad^3 + 108d^4$ = *Sum* of these *Prod.*
To which add $a^4 + 4d^4$ $\quad (= a^{3+1} + nd^{3+1})$

Sum is $a^4 + 24a^2d^2 + 88ad^3 + 112d^4 \,(= a^{n+1} + nd^{n+1} + z)$

From $\overline{l+d}|^{n+1} = \overline{a+4d}|^4 = a^4 + 16a^3d + 96a^2d^2 + 256ad^3 + 256d^4$
Subd. $a^{n+1} + nd^{n+1} + z = a^4 \quad * \quad + 24a^2d^2 + 88ad^3 + 112d^4$
Then

$$\overline{l+d}|^{n+1} - \overline{a^{n+1}+nd^{n+1}+z} = 16a^3d + 72a^2d^2 + 168ad^3 + 144d^4$$

And $\dfrac{16a^3d + 72a^2d^2 + 168ad^3 + 144d^4}{(m=)\ 4d} = 4a^3 + 18a^2d$
$+ 42ad^2 + 36d^3$, *the Sum of the Cubes of the given Terms.*
Because,

$$\text{The } \textit{Cube} \text{ of} \begin{cases} a \text{ is } a^3 \\ a+d \text{ is } a^3 + 3a^2d + 3ad^2 + d^3 \\ a+2d \text{ is } a^3 + 6a^2d + 12ad^2 + 8d^3 \\ a+3d \text{ is } a^3 + 9a^2d + 27ad^2 + 27d^3 \end{cases}$$

The *Sum* is $\quad 4a^3 + 18a^2d + 42ad^2 + 36d^3$,
the same with the *Quotient* found.

'Tis the same in any other *Series* for any other *Power.*

COROLLARY I.

Therefore, In a *Series* of *Laterals* beginning with 1, if $\overset{n}{s}$ be put for the *Sum* of the n *Power* thereof, Then

$$\overline{l+1}|^{n+1} - \overline{1+n+z} = m\overset{n}{s}. \text{ or,}$$

1. $\overline{l+1}|^2 - \overline{1+n} = 2\overset{1}{s}$
2. $\overline{l+1}|^3 - \overline{1+n+3\overset{1}{s}} = 3\overset{2}{s}$
3. $\overline{l+1}|^4 - \overline{1+n+4\overset{1}{s}+6\overset{2}{s}} = 4\overset{3}{s}$
4. $\overline{l+1}|^5 - \overline{1+n+5\overset{1}{s}+10\overset{2}{s}+10\overset{3}{s}} = 5\overset{4}{s}$
5. $\overline{l+1}|^6 - \overline{1+n+6\overset{1}{s}+15\overset{2}{s}+20\overset{3}{s}+15\overset{4}{s}} = 6\overset{5}{s}$

&c.

COROL.

COROLLARY 2.

Hence, when $a = 0$, then $n = l + 1$, And

$$1.\ \overline{l+1}|^2 - \overline{l+1} \quad = \quad ll + l = 2\overset{1}{s}$$

$$2.\ \overline{l+1}|^3 - \overline{l+1} + 3\frac{ll+l}{2} \quad = \frac{2l^3+3l^2+l}{2} = 3\overset{2}{s}$$

$$3.\ \overline{l+1}|^4 - \overline{l+1} + 6\frac{2l^3+3l^2+l}{6} + 4\frac{l^2+l}{2} = l^4+2l^3+l^2 = 4\overset{3}{s}$$

&c.

Therefore,

$$1.\ s = \frac{l^2+l}{2} = \frac{l+1}{2}l = \frac{n}{2}l.$$

$$2.\ \overset{2}{s} = \frac{2l^3+3l^2+l}{6} = \frac{l+1}{3}l^2 + \frac{l+1}{6l}l^2 = \frac{n}{3}l^2 + \frac{n}{6n-6}ll.$$

$$3.\ \overset{3}{s} = \frac{l^4+2l^3+l^2}{4} = \frac{l+1}{4}l^3 + \frac{l+1}{4l}l^3 = \frac{nl^3}{4} + \frac{nl^2}{4}.$$

&c.

COROLLARY 3.

But if the *Number of Terms* n be fuppofed *Infinite*, and g be put for the *greateft Term or Power*;

Then $\overset{n}{s} = \frac{ng}{n+1}$, therefore $1 : n+1 :: \overset{n}{s} : ng$.

This Proportion will hold, whether n be *Affirmative*, or *Negative*, *Whole*, *Fracted*, or *Surd Quantity*.

DEFINITION II.

THe Sums *of Numbers in a* Continued Arithmetic Proportion *from* Unity *are call'd* Figurate *or* Combinatory Numbers. *Thus*, If $1, 1+d, 1+2d, 1+3d, 1+4d, 1+5d, 1+6d$, &c. be an *Arithmetic Progression*; then

$1, 1+\overline{1+d}, 1+\overline{1+d}+\overline{1+2d}, 1+\overline{1+d}+\overline{1+2d}+\overline{1+3d}$ &c. are *Figurate Numbers.*

For the *Units* in each may be disposed into the *Form* of a *Regular Polygon*, whose *Number* of *Angles* is $d+2$, and whose *Side* (s) is equal to the *Number* of *Terms* that compose it.

Let f be the *Figurate Number*,
s its *Side*, or *Number* of *Terms* composing it.
a the *Number of Angles* ($=d+2$).

PROBLEM 2.

Case 1. Given s, d; Required f?

SOLUTION.

Since f is but the *Sum* of a *Series Arithmetically Proportional* beginning with 1, by this *Definition*;

Theref. $\frac{s+sl}{2}$ $\left(=\frac{s\times\overline{1+l}}{2}\right)=f$, by *Lemma* 1.

But $sd-d+1=l$ by *Lemma* 2.

Conseq. $f=\frac{2s+ssd-sd}{2}=s+\frac{\overline{ss-s}\times d}{2}$ by *Subst.*

Case 2. Given s, a; Required f?

SOLUTION.

Since $a=d+2$, or $d=a-2$, by this *Definition*;

Theref. $f=\frac{4+\overline{sa-a-2s}\times s}{2}$ by *Substitution.*

PROB.

PROBLEM. 3.

Case 1. Given f, d; Required s?

SOLUTION.

Since $f = \frac{2s + ssd - sd}{2} = s + \frac{ssd - sd}{2}$ by Prob. 2.

Theref. $ss + \frac{2s - sd}{d} = ss + s \times \frac{2 - d}{d} = \frac{2f}{d}$,

Conseq. $s = \frac{\overline{8fd + dd - 4d + 4}|^{\frac{1}{2}} + d - 2}{2d}$

Case 2. Given f, a; Required s?

SOLUTION.

Since $d = a - 2$, by this *Definition.*

Theref. $s = \frac{\overline{a^2 - 8a + 16 + 8af - 16f}|^{\frac{1}{2}} + a - 4}{2a - 4}$

DEFINITION III.

The Sums *of* { Polygonals / 1*st.* Pyramidals / 2*d.* Pyramidals / 3*d.* Pyramidals / *&c.* } *from* Unity *are call'd* { 1*st.* / 2*d.* / 3*d.* / 4*th.* / *&c.* } Pyramidals, *having their* Names *from their* Number *of* Sides.

COROLLARY.

Therefore in a *Rank* of { *Units* / *Laterals* / *Triangulars* / 1ft. *Pyramid* / &c. } their *Sums* are called { *Laterals* / *Triangulars* / 1ft *Pyramid.* / 2d *Pyramid.* / &c. } or *Figurates* of the { 2d / 3d / 4th / 5th / &c. } *Order.*

EXAMPLE.

Units	1 . 1 . 1 . 1 . 1 . 1 . 1 . 1 . 1
Laterals	1 . 2 . 3 . 4 . 5 . 6 . 7 . 8 . 9
Triangulars	1 . 3 . 6 . 10 . 15 . 21 . 28 . 36 . 45
1st. *Pyramid.*	1 . 4 . 10 . 20 . 35 . 56 . 84 . 120 . 165
2d. *Pyramid.*	1 . 5 . 15 . 35 . 70 . 126 . 210 . 330 . 495
&c.	

1. Here 'tis evident, that each *Figurate Number* is the *Aggregate* of the preceding *Series* so far.

2. That each *Figurate Number* is also equal to the *Sum* of the preceding *one*, and that above it.

PROBLEM 4.

To find the Sum *of such* Series, *or to find any particular* Figurate Number; *by having the* Side *or* Number *of* Terms *composing it given.*

1. In *Triangulars* or *Figurates* of the 3d. Order.

Since every *Triangular Number* is $\frac{ss+s}{2}$, by *Definit.* 3.

Theref. $\frac{AA+A}{2}$, $\frac{AA+A}{2}$, $\frac{aa+a}{2}$, &c. is a *Series* of *Triangulars*, or *Figurates* of the 3d. *Order*.

But $A + A + a$, &c. $= \frac{ss+s}{2}$

And $AA + AA + aa$, &c. $= \frac{s^3+s^2}{3} + \frac{s^3+s^2}{6s}$

} by C. 2. P. 1.

Theref.

Theref. $\frac{AA+\mathit{AA}+aa+A+\mathit{A}+a\&c.}{2}=\frac{s^3+3s^2+2s}{6}$

is the *Sum* of any *Series* of *Figurates* of the 3d. *Order*; or is a *Figurate Number* of the 4th *Order*, whofe *fide* is *s*.

II. *In* Figurate *Numbers of the 4th* Order.

Since every *Figurate* of the 4th. *Order* is $\frac{s^3+3s^2+2s}{6}$ by the preceding *Problem*; Therefore the *Series* is

$\frac{A^3+3A^2+2A}{6}, \frac{\mathit{A}^3+3\mathit{A}^2+2\mathit{A}}{6}, \frac{a^3+3a^2+2a}{6}$ &c.

But $\left\{\begin{array}{ll} 2A+2\mathit{A}+2a, & \&c. = ss+s \\ 3A^2+3\mathit{A}^2+3a^2, & \&c. = s^3+s^2+\frac{s^3+s^2}{2s} \\ A^3+\mathit{A}^3+a^3, & \&c. = \frac{s^4+s^3}{4}+\frac{s^4+s^3}{4s} \end{array}\right\}$ by Cor. 2. Prob. 1.

And $\frac{1}{6}$ of the *Sum* muft be $\frac{s^4+6s^3+11s^2+6s}{24}$

which is the Sum of a *Series* of *Figurates* of the 4th *Order*; or a *Figurate* of the 5th *Order*, whofe *fide* is *s*.

Therefore,

$$\left\{\begin{array}{ll} \frac{s^2+s}{2} & =\frac{s+0}{1}\times\frac{s+1}{2} \\ \frac{s^3+3s^2+2s}{6} & =\frac{s+0}{1}\times\frac{s+1}{2}\times\frac{s+2}{3} \\ \frac{s^4+6s^3+11s^2+6s}{24} & =\frac{s+0}{1}\times\frac{s+1}{2}\times\frac{s+2}{3}\times\frac{s+3}{4} \\ \&c. & \&c. \end{array}\right\}$$

is a *Figurate Number* of the $\left\{\begin{array}{l}\text{3d.}\\ \text{4th.}\\ \text{5th.}\end{array}\right\}$ *Order*; or the *Sum* of a *Series* of the *Figurates* $\left\{\begin{array}{l}\text{2d.}\\ \text{3d.}\\ \text{4th.}\end{array}\right\}$ *Order*

E X-

EXAMPLE.

Given, *The Number of Terms, suppose* 8, *in a Series of Figurates of the* 4*th Order*;

Required, *The Sum of that Series*; *or the* 8*th. Figurate of the* 5*th. Order.*

Substitute 8 in the room of *s*, then by the *Rule*,

$$\frac{8}{1}\times\frac{8+1}{1}\times\frac{8+2}{3}\times\frac{8+3}{4}=330,$$ required.

SCHOLIUM.

From what has been here said, we may easily Investigate that excellent *Theorem* of the Illustrious Mr. *Newton*, for *Raising a Binomial, to any given Power.*

For let any *Binomial* $(a+x)$ be *raised* to any *Power*, whose *Index* suppose n, (representing any Number, *Affirmative* or *Negative*, *Integer* or *Fraction*; The several *Powers* of that *Binomial* are,

1. $a+1x$
2. $a^2+2ax+1x^2$
3. $a^3+3a^2x+3ax^2+1x^3$
4. $a^4+4a^3x+6a^2x^2+4ax^3+1x^4$
5. $a^5+5a^4x+10a^3x^2+10a^2x^3+5ax^4+1x^5$
6. $a^6+6a^5x+15a^4x^2+20a^3x^3+15a^2x^4+6ax^5+1x^6$.

&c.

Here 'tis evident at sight, that the *Uncia* of the $\left\{\begin{matrix}\text{1st.}\\ \text{2d.}\\ \text{3d.}\\ \text{4th}\\ \text{\&c.}\end{matrix}\right\}$ *Term* is a *Figurate Number* of the $\left\{\begin{matrix}\text{1st.}\\ \text{2d.}\\ \text{3d.}\\ \text{4th.}\\ \text{\&c.}\end{matrix}\right\}$ *Order*, whose *Place* or *Side* is expressed by $\left\{\begin{matrix}1\\ n-0\\ n-1\\ n-2\\ \text{\&c.}\end{matrix}\right\}$;

Which

Which being Substituted in the room of s, in the preceding *Theorems* (for finding such *Figurate Numbers*) shall give.

$$\left.\begin{array}{l} 1 \\ 1\times\frac{n-0}{1} \\ 1\times\frac{n-1}{1}\times\frac{n-0}{2} \\ 1\times\frac{n-2}{1}\times\frac{n-1}{2}\times\frac{n-0}{3} \\ \text{\&c.} \end{array}\right\} \text{ or } \left\{\begin{array}{l} 1 \\ 1\times\frac{n-0}{1} \\ 1\times\frac{n-0}{1}\times\frac{n-1}{2} \\ 1\times\frac{n-0}{1}\times\frac{n-1}{2}\times\frac{n-2}{3} \\ \text{\&c.} \end{array}\right.$$

the *Unciæ* or *Coefficients* of the 1st. 2d. 3d. 4th. &c. *Term.*

Therefore, the *Unciæ* of any *Binomial* $(a+x)$ rais'd to the *Power* whose *Index* is n, will be

$$1\times\frac{n-0}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}\times\frac{n-3}{4}\times\frac{n-4}{5}\times\frac{n-5}{6}, \text{\&c. } i.e.$$

$$1.\ 1\times\frac{n-0}{1}.\ 1\times\frac{n-0}{1}\times\frac{n-1}{2}.\ 1\times\frac{n-0}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}, \text{ \&c.}$$

respectively. Therefore,

$$\overline{a+x}|^{n} = a^{n}$$
$$+\frac{n-0}{1}a^{n-1}x$$
$$+\frac{n-0}{1}\times\frac{n-1}{2}a^{n-2}x^{2}$$
$$+\frac{n-0}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}a^{n-3}x^{3}$$
$$+\frac{n-0}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}\times\frac{n-3}{4}a^{n-4}x^{4}$$

&c.

Or putting $q = \frac{x}{a}$, then $a + x = a + a\frac{x}{a} = a + aq = a \times \overline{1 + q}$

And $\overline{a + aq}|^n = a^n$

$+ \frac{n-0}{1} a^n q$

$+ \frac{n-0}{1} \times \frac{n-1}{2} a^n q^2$

$+ \frac{n-0}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} a^n q^3$

$+ \frac{n-0}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} a^n q^4$

&c.

Also putting A = 1st. *Term*, B = 2d. C = 3d. D = 4th, &c. it will be,

$\overline{a + aq}|^n = a^n + \frac{n}{1} Aq + \frac{n-1}{2} Bq + \frac{n-2}{3} Cq + \frac{n-3}{4} Dq$, &c.

Or if the *Index* be $\frac{m}{n}$, then

$\overline{a + aq}|^{\frac{m}{n}} = a^{\frac{m}{n}} + \frac{m}{n} Aq + \frac{m-n}{2n} Bq + \frac{m-2n}{3n} Cq + \frac{m-3n}{4n} Dq$, &c.

Note, That there are several other ways of *Investigating* this *Important Theorem*, yet none here would be so appofite as this.

And we have insisted the more thereon, because its extensive *Use* is almost *Infinite*; since 'tis not for the *Discovery* of one *Particular Case* alone that it serves, 'tis not only *Involution*, *Evolution*, *Division by Powers*, or *Radical Quantities*, and the like, as well in *Numbers* as *Species* that it performs; But it even Comprehends the *Method of Indivisibles*, the *Arithmetic of Infinites*, the *Doctrine of Series*: and in a word, there is scarce any *Inquiry* so Sublime and Intricate, or any *Improvement* so Eminent and

and Considerable, in *Pure Mathematicks*, but by a *Prudent application* of this *Theorem*, may easily be exhibited and deduced; and that by a *General* and *Direct Calculation*, in as Perfect a manner as the Nature of the *Thing* will admit; a few of which we shall Instance.

I. *To raise any* Tri-nomial, Quadri-nomial, *&c.* *or* Infinito-nomial *to any given* Power.

Suppose the *Infinito-nomial* $a + bz + cz^2 + dz^3$, &c. was to be *raised* to the *Power* whose *Index* is n.

Put $\overline{bz + cz^2 + dz^3, \&c.}|^1 = x^1$ } in the *Binom. Theor.*
And $\overline{bz + cz^2 + dz^3, \&c.}|^2 = x^2$
&c. Then we have this *Theorem*.

$$\overline{a + bz + cz^2 + dz^3, \&c.}|^n = a^n + \frac{n}{1}a^{n-1} \times \overline{bz + cz^2 + dz^3 \&c.}|^1 + \frac{n}{1} \times \frac{n-1}{2}a^{n-2} \times \overline{bz + cz^2 + dz^3 \&c.}|^2 + \&c.$$

that is

$$\overline{a + bz + cz^2 + dz^3, \&c.}|^n = a^n + \frac{n}{1}a^{n-1}bz + \frac{n}{1} \times \frac{n-1}{2}a^{n-2}b^2z^2 + \frac{n}{1}a^{n-1}c + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3}a^{n-3}b^3z^3 + \frac{n}{1} \times \frac{n-1}{1}a^{n-2}bc + \frac{n}{1}a^{n-1}d + \&c.$$

And Multiplying both *Sides* by z^n, 'twill be

$$\overline{az+bz^2+cz^3+dz^4, \&c.}|^n = a^n z^n$$
$$+\frac{n}{1}a^{n-1}bz^{n+1}$$
$$+\frac{n}{1}\times\frac{n-1}{2}a^{n-2}b^2z^{n+2}$$
$$+\frac{n}{1}a^{n-1}c$$
$$+\frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}a^{n-3}b^3z^{n+3}$$
$$+\frac{n}{1}\times\frac{n-1}{1}a^{n-2}bc$$
$$+\frac{n}{1}a^{n-1}d$$

&c. as express'd by the Author thereof, the Excellent Analyst Mr. *Ab. de Moivre*, (in *Philos. Trans.* N. 230.)

Here 'tis manifest, that

1. *All the Products, that can be made so as to have the Sum of the Exponents of the Letters composing them equal to some Index of the Power of z, must belong to* [illegible] *Power.*

[illegible]. *The Number, which expresses how many ways the Let-*[illegible] *of each Product may be changed, must be prefix'd to* [illegible] *Product.*

But

But by putting A = 1st. *Term*, B = 2d, C = 3d, D = 4th. &c. then will

$\overline{a + bz + cz^2 + dz^3 \&c.}|^n$ be found equal to

$$a^n$$

$$+ \frac{1nAb}{1a} z$$

$$+ \frac{2nAc + \overline{1n-1}Bb}{2a} z^2$$

$$+ \frac{3nAd + \overline{2n-1}Bc + \overline{1n-2}Cb}{3a} z^3$$

$$+ \frac{4nAe + \overline{3n-1}Bd + \overline{2n-2}Cc + \overline{1n-3}Db}{4a} z^4$$

$$+ \frac{5nAf + \overline{4n-1}Be + \overline{3n-2}Cd + \overline{2n-3}De + \overline{1n-4}Eb}{5a} z^5$$

&c. where the *Law of Continuation* is visible; but the *Application to Practice* is somewhat more difficult than the former.

II. *Of the Nature and Construction of* Logarithms.

From hence also, *the Celebrated Mathematician Mr.* Halley, *Savilian Professor of Geometry in* Oxford, among the several admirable Discoveries, and happy Advances, in *Useful Learning*, which he obliges the World with, has (in *Philos. Trans.* N. 216.) drawn a very curious Method for *Constructing Logarithms*, not only comprehending all the Improvements that *Mercator*, *Gregory*, and others have made by the Help of *Geometric Figures*, but shewing with great Accuracy, from the *Common Properties of Numbers*, (as most Natural and Agreeable in *Things* purely *Arithmetical*,) how the *Logarithms* may be produced to *any* desired *Number of Places*, with far more Ease and Expedition than by any Method known before.

Since

Since these *Logarithms* (Invented by the Lord *Naper*) as Improved by Mr. *Briggs*, are one of the most useful Discoveries in *Arithmetic*, and the *Demonstration* of this *Method of Constructing* them follows properly in this Place, and is so worthy the *Reader's* Knowlege; we have here inserted a Specimen of their *Nature* and *Construction*, according to Mr. *Halley*'s Method.

1. Supposing an *Infinite Number n* of equal *Ratio's* or *Ratiunculæ* in a continued *Scale of Proportionals* between the two *Terms* of any *Ratio*, as between 1 and 1 + x or $\overline{1+\dot{x}}|^n$, then $1+\dot{x}$ will be the *First Mean* of *Root* of the *Infinite Power* $\overline{1+\dot{x}}|^n$, and let $\dot{x}$ be a *Ratiuncula*, or *Fluxion* of the *Ratio* of 1 to 1 + x, then

$$\&c. \quad -2 \quad -1 \quad 0 \quad +1 \quad +2 \quad \&c. \quad +n$$

$$\&c. \; \frac{1}{\overline{1+\dot{x}}|^2},\; \frac{1}{\overline{1+\dot{x}}|^1},\; 1,\; \frac{\overline{1+\dot{x}}|^1}{1},\; \frac{\overline{1+\dot{x}}|^2}{1},\; \&c. \; \frac{\overline{1+\dot{x}}|^n}{1}$$

where 'tis evident, that any *Index* expresses the *Number* of *Ratiunculæ* contain'd in the *Ratio* of 1 to such *Term*.

2. Hence, we may value *Ratio's* by the *Number* of *Ratiunculæ* contain'd in each, and may consider them as *Quantitates sui generis*, beginning from the *Ratio* of 1 to 1 = 0, being *Affirmative*, or *Negative*, according to their *Increase* or *Decrease*, *above* or *below Unity*.

So that *Ratio's* may be to one another as the *Number* of like and equal *Ratiunculæ* contain'd between their *Terms*; and their *Duplicate*, *Triplicate*, &c. contains *Twice*, *Thrice*, &c. that *Number*.

3. And

3. And the Number of *Ratiunculæ* between 1 and any *Number* or the *Value* (i. e. the *Exponent*) of the *Ratio* of *Unity* to any *Number*, is called the *Logarithm* of that *Number*, as it's *Etymon* λόγων ἀριθμὸς very properly Imports.

Thus suppose between 1 and 10 an *Infinite Number of mean Proportionals*, expressed by 10000 &c. *in infinitum*.

Then between 1 and 2, 1 and 3, 1 and 4, 1 and 10; There will be 3010 &c. 4771 &c. 6021 &c. 10000 &c. which are the *Logarithms* of 2, 3, 4, 10, or rather the *Logarithms* of the *Ratio* of 1 to 2, 1 to 3, 1 to 4, 1 to 10.

So if the *Ratio* of 1 to 10 } contains { 1000 } &c. *Ratiunculæ*
That of its { *Duplicate* / *Triplicate* / &c. } contains { 2000 / 3000 / &c. } &c. *Ratiunculæ*

Therefore, *Logarithms* or the *Values* of *Ratio's* are in an *Arithmetic Progression*.

4. But because any *Infinite Number of means* may be taken between the *Terms* of any *Ratio*, provided the same *Proportion* be every where observed; therefore $n\dot{x}$ may as well be put for the *Logarithm* of 1 to $1+x$, *i. e.* the *Sum of the Ratiunculæ* may as well be the *Index* as the *Number of them*; then,

$$0. \quad 1\dot{x}, \quad 2\dot{x}, \quad 3\dot{x}, \quad 4\dot{x}, \text{ \&c. } n\dot{x}.$$

$$1, \ \overline{1+\dot{x}}|^{1}, \ \overline{1+\dot{x}}|^{2}, \ \overline{1+\dot{x}}|^{3}, \ \overline{1+\dot{x}}|^{4}, \text{ \&c. } 1+x$$

Therefore $\overline{1+x}|^{\frac{1}{n}} - 1 = \dot{x}$ (the *Ratiuncula*)

And $\overline{\overline{1+x}|^{\frac{1}{n}} - 1} \times n = n\dot{x} = \text{L}, 1+x$

Therefore

Theref. $\overline{\overline{1+9}|^{\frac{1}{1000\,\&c.}} - 1} \times 1000,\&c. = 1000\&c. \times \dot{x} = L,10.$

And $\overline{\overline{1+1}|^{\frac{1}{3010\,\&c.}} - 1} \times 3010,\&c. = 3010\&c. \times \dot{x} = L,2.$

Whence, $L,1+x : L,1+z :: N\dot{x} : n\dot{z}$.

And these *Ratiunculæ* being hitherto considered as having the *same Magnitude* in all *Ratio's* ;

Therefore $L,1+x : L,1+z :: N : n$. That is *The Logarithms of Ratio's are as the Number of Ratiunculæ contain'd between their Terms :*

But if the *Number of Ratiunculæ* be supposed the *same* in all *Ratio's*,

Then $L,1+x : L,1+z :: \dot{x} : \dot{z}$, that is *The Logarithm of each Ratio will be as the Fluxion thereof.*

Therefore the *Logarithm* of any *Number* is found by *taking the Difference between Unity and the Infinite Root of that Number.*

Hence 'tis evident that there may be as many different *Scales of Logarithms*, as there are assumed different *Infinite Indices* (n) of the *Power* whose *Root* is sought.

If $n = \left\{ \begin{array}{l} 10000 \text{ \&c. the Lord } \textit{Naper's} \\ 23025 \text{ \&c. Mr. } \textit{Briggs's} \end{array} \right\}$ *Logarithm* will be produced.

Therefore, put $\overline{1 \pm x}|^{\frac{1}{n}}$ for the *Infinite Power* to be resolv'd ; Then,

The *Coefficients* being $1 \times \frac{\frac{1}{n} - 0}{1} \times \frac{\frac{1}{n} - 1}{2} \times \frac{\frac{1}{n} - 2}{3} \times \frac{\frac{1}{n} - 3}{4}$ &c. when n is *Finite*, by the former *Rules*.

But

But $\frac{\frac{1}{n}-0}{1}$ ——— $\times$ $1 = \quad +\frac{1}{n}$

$\frac{\frac{1}{n}-1}{2}\left(\text{or } \frac{1}{2n}-\frac{1}{2}\right) \times +\frac{1}{n} = \frac{1}{2nn}-\frac{1}{2n}$

$\frac{\frac{1}{n}-2}{3}\left(\text{or } \frac{1}{3n}-\frac{2}{3}\right) \times -\frac{1}{2n} = \frac{1}{6nn}+\frac{1}{3n}$

$\frac{\frac{1}{n}-3}{4}\left(\text{or } \frac{1}{4n}-\frac{3}{4}\right) \times +\frac{1}{3n} = \frac{1}{12nn}-\frac{1}{4n}$

And nn being *Infinitly Infinite*, therefore that which is divided thereby must vanish: Consequently the *Coefficients* will be

$+\frac{1}{n}, -\frac{1}{2n}, +\frac{1}{3n}, -\frac{1}{4n}, +\frac{1}{5n}$, &c. And

$\overline{1\pm x}|^{\frac{1}{n}} = 1 \pm \frac{1}{n}x - \frac{1}{2n}x^2 \pm \frac{1}{3n}x^3 - \frac{1}{4n}x^4 \pm$ &c. Hence

$\frac{1}{n} \times \overline{x \mp \frac{1}{2}x^2 + \frac{1}{3}x^3 \mp \frac{1}{4}x^4 + \&c.} = 1 \backsim \overline{1\pm x}|^{\frac{1}{n}} = x = L\,\overline{1+x}$

And since the *Infinite Index* (n), may be assumed at pleasure, the several *Scales of Logarithms* to such *Indices* will be as $\frac{1}{n}$, or *Reciprocally* as such *Indices*. Therefore, if $n = 10000$, &c. as in *Neper*'s Logarithms; then $x \mp \frac{1}{2}x^2 + \frac{1}{3}x^3 \mp \frac{1}{4}x^4 + \frac{1}{5}x^5$, &c. $= x = L,\ \overline{1\pm x}$.

But if neither of the *Terms* of the *Ratio* be 1, they must be reduced into such, wherein one of the *Terms* may be 1: Thus,

If the *Least* be a, the *Greatest* b; let $s = b+a$, $d = b-a$;

Then $\begin{cases} a : b :: 1 : 1+x = \frac{b}{a} \therefore x = \frac{b}{a} - 1 = \frac{b-a}{a} = \frac{d}{a} \\ b : a :: 1 : 1-x = \frac{a}{b} \therefore x = 1 - \frac{a}{b} = \frac{b-a}{b} = \frac{d}{b} \end{cases}$

Therefore the *Logarithm* of the *Ratio* contain'd between a and b, may be *doubly* expreſs'd.

Now ſuppoſe the *Ratio* of a to b, was divided into that of a to $\frac{1}{2}s$, and of $\frac{1}{2}s$ to b.

$$\text{Then L,}\ \frac{a}{\frac{1}{2}s}+\text{L,}\ \frac{\frac{1}{2}s}{b}=\text{L,}\ \frac{a}{b};\ \text{for}\ \frac{a}{\frac{1}{2}s}\times\frac{\frac{1}{2}s}{b}=\frac{\frac{1}{2}sa}{\frac{1}{2}sb}=\frac{a}{b}$$

$$\text{Or L,}\ \frac{\frac{1}{2}s}{a}+\text{L,}\ \frac{b}{\frac{1}{2}s}=\text{L,}\ \frac{b}{a};\ \text{for}\ \frac{\frac{1}{2}s}{a}\times\frac{b}{\frac{1}{2}s}=\frac{\frac{1}{2}sb}{\frac{1}{2}sa}=\frac{b}{a}$$

$$\text{And}\left\{\begin{array}{l}\frac{1}{2}s:a::1:1-x=\frac{a}{\frac{1}{2}s}\\ \frac{1}{2}s:b::1:1+x=\frac{b}{\frac{1}{2}s}\end{array}\right\}\ \|\ x\ \therefore\left\{\begin{array}{l}1-\frac{a}{\frac{1}{2}s}=\frac{\frac{1}{2}s-a}{\frac{1}{2}s}\\ \frac{b}{\frac{1}{2}s}-1=\frac{b-\frac{1}{2}s}{\frac{1}{2}s}\end{array}\right\}=\frac{d}{s}$$

for both *Ratios*; therefore

$$\left.\begin{array}{l}\frac{1}{n}\times\overline{\frac{d}{s}+\frac{d^2}{2s^2}+\frac{d^3}{3s^3}+\frac{d^4}{4s^4}+\frac{d^5}{5s^5}},\text{\&c.}=\text{A}=\text{L,}\\ \frac{1}{n}\times\overline{\frac{d}{s}-\frac{d^2}{2s^2}+\frac{d^3}{3s^3}-\frac{d^4}{4s^4}+\frac{d^5}{5s^5}},\text{\&c.}=\text{B}=\text{L,}\end{array}\right\}\text{of Rat. betw.}\left\{\begin{array}{l}a\ \&\ \frac{1}{2}s\\ \frac{1}{2}s\ \&\ b\end{array}\right\}$$

RULE I.

$$\frac{1}{n}\times\overline{\frac{2d}{s}*+\frac{2d^3}{3s^3}*+\frac{2d^5}{5s^5}},\text{\&c.}=\text{A}+\text{B}=\text{L, } \textit{Rat.}\text{ of } a \text{ to } b.$$

$$\text{And A} \backsim \text{B}=\text{L,}\ \frac{\frac{1}{2}s}{b})\ \frac{a}{\frac{1}{2}s}\ (=\frac{ab}{\frac{1}{4}ss}\ \text{or}$$

$$\frac{1}{n}\times\overline{*\frac{2d^2}{2s^2}*+\frac{2d^4}{4s^4}*+\frac{2d^6}{6s^6}\ \text{\&c.}}=\text{L,}\ \frac{\sqrt[2]{ab}}{\frac{1}{2}s}\times\frac{\sqrt[2]{ab}}{\frac{1}{2}s}$$

$$\therefore\frac{1}{n}\times\overline{\frac{d^2}{2s^2}+\frac{d^4}{4s^4}+\frac{d^6}{6s^6}+\frac{d^8}{8s^8}},\ \text{\&c.}=\text{L, of the } \textit{Ratio}$$

of the *Geometrical* to the *Arithmetical mean* between (a) and (b).

But

But the *Difference* of the *Terms* of the *Ratio*, *i. e.* $\frac{1}{4}ss - ab$, or $\overline{\frac{1}{4}a^2 + \frac{1}{2}ab + \frac{1}{4}b^2} - ab = \frac{1}{4}a^2 - \frac{1}{2}ab + \frac{1}{4}b^2 = \overline{\frac{1}{2}a - \frac{1}{2}b}|^2 = \frac{1}{4}d^2 = 1$ (in this *Cafe*): And putting $\frac{1}{4}ss + ab = y$, ∴ (fince $y = s$, and $d = 1$) it follows, that $\frac{1}{n} \times \overline{\frac{2}{y} + \frac{2}{3y^3} + \frac{2}{5y^5} + \frac{2}{7y^7}}$ &c. $= L$, of the *Ratio* between $\frac{1}{4}ss$ and ab, by *Rule* 1. Whence

RULE 2.

$$\frac{1}{n} \times \overline{\frac{1}{y} + \frac{1}{3y^3} + \frac{1}{5y^5} + \frac{1}{7y^7}} + \text{\&c.} = L, \text{ of the}$$

Ratio between $\frac{1}{2}s$ and $\sqrt[2]{ab}$. Which *Rule* is of Excellent ufe for finding the *Logarithms* of *Prime Numbers*, having the *Logarithms* of the adjoyning *Numbers* given.

In making *Briggs*'s Logarithms, the *Index* (n) muft be 2,3025850, &c. as was hinted before;

For if the *Index* n be 100000000 &c. *in infinitum*,

The Logarithm of 10 will be 2,3025850, &c. as in *Naper*'s;

But that the Logarithm of 10 may be 1000000, &c. as in *Briggs*'s,

The *Index* n muft be 2, 3025850, &c.

And this *Index*, *i. e.* *Naper*'s Logarithm of 10, may be eafily found feveral ways, either by the Number 10 it felf, or by its *Component Parts*: But we fhall inftance only this *way*.

Since $10 = 2 \times 2 \times 2 \times 1\frac{1}{4}$ ∵ $L, 10 = 3L, 2 + L, 1\frac{1}{4}$; therefore the *Log.* of 2, and the *Log.* of $1\frac{1}{4}$ muft be found to the *Index* 100000, &c.

And *Nap.* L, 2 is $\frac{2}{3} + \overline{\frac{1}{3} \times \frac{2}{3^3}} + \overline{\frac{1}{5} \times \frac{2}{3^5}} + \overline{\frac{1}{7} \times \frac{2}{3^7}}$ &c. *R.* 1.

$\therefore$ 3 L, 2 $= \frac{6}{3} + \overline{\frac{1}{3} \times \frac{6}{3^3}} + \overline{\frac{1}{5} \times \frac{6}{3^5}} + \overline{\frac{1}{7} \times \frac{6}{3^7}}$ &c. by *Mult.*

Alſo *Nap.* L, $1\frac{1}{4}$ is $\frac{2}{9} + \overline{\frac{1}{3} \times \frac{2}{9^3}} + \overline{\frac{1}{5} \times \frac{2}{9^5}} + \overline{\frac{1}{7} \times \frac{2}{9^7}}$ &c. *R.* 1

But $\frac{2}{9} = \frac{6}{3^3}$, $\frac{2}{9^3} = \frac{6}{3^7}$, $\frac{2}{9^5} = \frac{6}{3^{11}}$, &c. Therefore.

The Operation ſtands thus:

$\frac{6}{3} =$	2, ...	$\frac{1}{1}$ 2		2, ...
$\frac{1}{9}$ 2 = A	2222222222222	$\frac{1}{3}$ A + $\frac{1}{1}$ A		2962962962962
$\frac{1}{9}$ A = B	246913580246	$\frac{1}{5}$ B		49382716049
$\frac{1}{9}$ B = C	27434842249	$\frac{1}{7}$ C + $\frac{1}{3}$ C		13064210595
$\frac{1}{9}$ C = D	3048311805	$\frac{1}{9}$ D		338701756
$\frac{1}{9}$ D = E	338701756	$\frac{1}{11}$ E + $\frac{1}{5}$ E		98531420
$\frac{1}{9}$ E = F	37633529	$\frac{1}{13}$ F		2894887
$\frac{1}{9}$ F = G	4181503	$\frac{1}{15}$ G + $\frac{1}{7}$ G		876124
$\frac{1}{9}$ G = H	464612	$\frac{1}{17}$ H		27330
$\frac{1}{9}$ H = I	51624	$\frac{1}{19}$ I + $\frac{1}{9}$ I		8454
$\frac{1}{9}$ I = K	5763	$\frac{1}{21}$ K		273
$\frac{1}{9}$ K = L	637	$\frac{1}{23}$ L + $\frac{1}{11}$ L		85
$\frac{1}{9}$ L = M	71	$\frac{1}{25}$ M		3
Therefore *Naper*'s Log. of 10, is				2,302585092994o

Or the *Index* (*n*) for *Brigg*'s Scale of Logarithms ; which, by continuing the the Operation, is found to be 2.30258509299404568401799145468436420760110148 8628772976033328, *&c.* therefore $\frac{1}{n}$ will be 0.43429448190325182765112891891660508229439700 5803666566114454, *&c.*

EXAM-

EXAMPLE I.

To find *Briggs*'s Logarithm of 2, only to 10 Places.

Note, That the *Index* must be assumed of a *Figure* or *two* more than the intended *Logarithm* is to have; therefore, in this *Example*, $\frac{1}{n} = 0,434294481903 = \beta$, $d = 1$, and $s = 3$.

$$\text{And } \beta \times \overline{\frac{2}{3} + \frac{1}{3} \times \frac{2}{3^3} + \frac{1}{5} \times \frac{2}{3^5} + \frac{1}{7} \times \frac{2}{3^7} \&c.} = L,2. \; R, 1,$$

$$\text{Or } \frac{\beta}{3} + \frac{1}{3} \times \frac{\beta}{3^3} + \frac{1}{5} \times \frac{\beta}{3^5} + \frac{1}{7} \times \frac{\beta}{3^7} \&c. = \tfrac{1}{2} L,2.$$

The Operation *stands thus:*

$\frac{1}{3}\beta = A$	,144764827301	$\frac{1}{1} A =$	,144764827301
$\frac{1}{9} A = B$	16084980811	$\frac{1}{3} B =$	5361660270
$\frac{1}{9} B = C$	1787220090	$\frac{1}{5} C =$	357444018
$\frac{1}{9} C = D$	198580010	$\frac{1}{7} D =$	28368572
$\frac{1}{9} D = E$	22064445	$\frac{1}{9} E =$	2451605
$\frac{1}{9} E = F$	2451605	$\frac{1}{11} F =$	222873
$\frac{1}{9} F = G$	272400	$\frac{1}{13} G =$	20953
$\frac{1}{9} G = H$	30266	$\frac{1}{15} H =$	2017
$\frac{1}{9} H = I$	3362	$\frac{1}{17} I =$	197
$\frac{1}{9} I = K$	373	$\frac{1}{19} K =$	19
$\frac{1}{9} K = L$	41	$\frac{1}{21} L =$	1
	Sum $= \frac{1}{2}$ L, 2		= 0,150514997826
			× 2
	Theref. the *Logarithm* of 2 is		0,30102999956,52

But the same *Logarithm* may yet be obtained much easier and sooner from this Consideration, *viz.*

That

That $\overline{\frac{1}{2}}\Big|^{10} = \frac{1}{1024}$, and $\frac{1000}{1024} \times \frac{1}{1000} = \frac{1}{1024}$;

Theref. $\frac{L,\frac{1000}{1024} + L,\frac{1}{1000}}{10} = \frac{L,\frac{1}{1024}}{10} = L,1 \text{ to } 2, = L,2.$

But $L,\frac{1000}{1024} = L,\frac{125}{128} = \beta \times \overline{\frac{2}{1} \times \frac{3}{253} + \frac{2}{3} \times \frac{3^3}{253^3} + \frac{2}{5} \times \frac{3^5}{253^5}}$ &c.

i.e. $L,\frac{125}{128} = \frac{2\beta d}{s} + \frac{2\beta d^3}{3s^3} + \frac{2\beta d^5}{5s^5} + \frac{2\beta d^7}{7s^7}$ &c. by R. 1. Let $\frac{d}{s} = q$.

$\frac{1}{1} \times \frac{1}{s} \times 2\beta d$	$= A =$	0.010299473 87912
$\frac{1}{3} \times qq \times A$	$= B \mp$	48271995
$\frac{3}{5} \times qq \times B$	$= C =$	4072
Sum $L,\frac{1000}{1024}$ $(= L,\frac{125}{128})$	$=$	0.010299956 63980
Add $L,\frac{1}{1000}$	$=$	3,000000000000000

And $\frac{1}{10}$ of that *Sum* is 0,301029995663980 $= L,2.$

EXAMPLE 2.

To find *Briggs*'s Logarithm of 3.

Here $d = 2$, $s = 4$, therefore $\frac{d}{s} = \frac{1}{2}$, conseq. by R. 1.

$L,3 = \beta \times \overline{\frac{2}{1} \times \frac{1}{2} + \frac{2}{3} \times \frac{1}{8} + \frac{2}{5} \times \frac{1}{32} + \frac{2}{7} \times \frac{1}{128}}$, &c.

$\frac{1}{2}L,3 = \frac{\beta}{2} + \frac{1}{3} \times \frac{\beta}{8} + \frac{1}{5} \times \frac{\beta}{32} + \frac{1}{7} \times \frac{\beta}{128}$, &c. whence the several *Terms* may be readily found by a *Continual Division* of $\frac{\beta}{2}$ by 4, and those again by the *Indices* of the *odd Powers*, each respectively, and *twice* the *Sum* of the *Quotes* will be the *Logarithm* of 3 sought.

But

But the *Logarithm* of 3, is found abundantly sooner, by *Rule* 2. The *adjoyning Numbers* being 2 $(=a)$ and 4 $(=b)$

And $\left\{ \begin{array}{l} \frac{1}{2}s = 3 \\ \sqrt[2]{ab} = \sqrt{4 \times 2} \end{array} \right\} = \left\{ \begin{array}{l} \textit{Arith.} \\ \textit{Geom.} \end{array} \right\}$ *mean* betw. 2 and 4.

And $\frac{1}{4}ss = 9$, and $\frac{1}{4}ss + ab = 9 + 8 = 17 = y$.

But $\frac{\sqrt{4 \times 2}}{\frac{1}{2}s} \times \frac{1}{\sqrt{4 \times 2}} = \frac{1}{\frac{1}{2}s}$, therefore,

$$L, \frac{1}{\sqrt{4 \times 2}} + L, \frac{\sqrt{4 \times 2}}{\frac{1}{2}s} = L, \frac{1}{\frac{1}{2}s} = L, \text{ of } 1 \text{ to } 3, \text{ or } L, 3.$$

$$\text{The } L, \frac{\sqrt{4 \times 2}}{\frac{1}{2}s} = \frac{\beta}{17} + \frac{1}{3} \times \frac{\beta}{17^3} + \frac{1}{5} \times \frac{\beta}{17^5} + \frac{1}{7} \times \frac{\beta}{17^7} \; \&c.$$

And $L, \frac{1}{\sqrt{4 \times 2}} = \frac{L,4 + L,2}{2}$	0,4515449934959
$\frac{1}{1} \times \frac{1}{y} \times \beta = A$	255467342296
$\frac{1}{3} \times \frac{1}{yy} \times A = B$	294656680
$\frac{3}{5} \times \frac{1}{yy} \times B = C$	611744
$\frac{5}{7} \times \frac{1}{yy} \times C = D$	1511
The Sum is the L, 3	0,4771212547190

And the same *Logarithm* may yet be very expeditiously found by means of the *Ratio* of 2^{15} to 5×3^8 (where $d = 37$, and $s = 65573$).

For $\frac{2^{15}}{5 \times 3^8} \times \frac{5}{2^{15}} = \frac{1}{3^8}$, Therefore,

$$\frac{L, \frac{2^{15}}{5 \times 3^8} + L, \frac{5}{2^{15}}}{8} \text{ or } \frac{L, \frac{2^{15}}{5 \times 3^8} + L, 2^{15} - L, 5}{8} = L, 3$$

But

But $L, 2^{15} - L, 5 = 3{,}81647993062$
And $\frac{1}{3} \times 2\beta d = 0{,}00049010708$ 1st *Step* of the *Series*.

Sum is $3{,}81697003770 = L, 3^8$
And $\frac{1}{8}$ thereof is $0.47712125471 = L, 3$.

Here the *first step* alone gives the *Logarithm* true to 11 *Places*.

Such *Contractions*, to expedite the *Operation*, may also be found for the other *Prime Numbers*, by means whereof the *Series* is made to *Converge* wonderfully quick: as the *Logarithm*

Of	by the *Ratio* of	to
7	$3 \times 2^5 \times 5^2$	7^4
11	$2 \times 7^2 \times 10^2$	$3^4 \times 11^2$
13	$7 \times 11 \times 5^2 \times 2^6$	$3^6 \times 13^2$
17	$13 \times 2^3 \times 5^2$	$3^2 \times 17^2$
19	$10^2 \times 17^2$	$3^2 \times 13^2 \times 19$
23	$10 \times 2^5 \times 3^4$	$7^2 \times 23^2$
29	$17 \times 23 \times 2^5 \times 7^2$	$3^6 \times 29^2$
31	$10 \times 17 \times 19 \times 2^2 \times 3^2$	$11^2 \times 31^2$
37	$3 \times 7 \times 17 \times 29 \times 2^4$	$11^2 \times 37^2$
41	$7 \times 10 \times 11 \times 17 \times 2^2$	$37^2 \times 41^2$
&c.	&c.	&c.

EXAMPLE 3.

To find *Briggs*'s Logarithm of the *Prime Number* 23 from the 2d. *Rule*; as perform'd by Mr. *Halley*, in the aforementioned *Philos. Trans.*

The *adjoyning Numbers* are 22 $(= a)$ and 24 $(= b)$ $\therefore d = 2$, $s = 46$, $\frac{1}{2}s = 23$, and $\frac{1}{4}ss + ab$ or $529 + 528 = 1057 = y$.

But

But

$$\frac{\sqrt{ab}}{\frac{1}{2}s} \times \frac{1}{\sqrt{ab}} = \frac{1}{\frac{1}{2}s} \therefore L, \frac{\sqrt{24 \times 22}}{\frac{1}{2}s} + L, \frac{1}{\sqrt{24 \times 23}} = L, \frac{1}{\frac{1}{2}s} = L, 23.$$

And $L, \frac{\sqrt{24 \times 22}}{\frac{1}{2}s} = \frac{\beta}{y} + \frac{\beta}{3y^3} + \frac{\beta}{5y^5} + \frac{\beta}{7y^7} + \frac{\beta}{9y^9}$ &c. by R. 1.

Since $2 \times 2 \times 2 \times 3 = 24$, and $2 \times 11 = 22$, therefore

$3L,2 + L,3 = L,24$, and $L,2 + L,11 L, = 22$. But $\frac{L,24 + L,22}{2}$

Or $L,\sqrt{ab} =$	1,361316961266906129450091726598 05
$\frac{1}{1} \times \frac{1}{y} \times \beta = A$	41087462810146814347315886368
$\frac{1}{3} \times \frac{1}{yy} \times A = B$	12258521544181829460074
$\frac{3}{5} \times \frac{1}{yy} \times B = C$	658323518437 6175
$\frac{5}{7} \times \frac{1}{yy} \times C = D$	4208829765
$\frac{7}{9} \times \frac{1}{yy} \times D = E$	2930
Sum $= L. 23 =$	1.361727836017592878867777 11225117

The *Ingenious* may for his Practice, with the same ease continue the *Logarithm* to any *Number of Places*, by taking the *Index* accordingly; tho the design of this *Treatise*, and the narrowness of the *Page*, determin'd our *Example* but to few *Places*; in the Explication of which, we have bin the more large, to the end that the *Logarithm* of any other *Incompoſite Number* may be made by the foregoing *Rules* without any farther Direction.

NOTE.

1. The *Logarithms* of *Compoſite Numbers* are found by *Adding of the Logarithms of their Factors*; Thus, the *Logarithm* of 6 is the *Sum* of the *Logarithms* of 2 and 3, for $2 \times 3 = 6$.

2. The *Logarithms* of the *Powers* of any *Number* are obtain'd by *Multiplying the Logarithm of that Number by the Index of the Power*; for these *Indices* are *Proportional* to those *Logarithms:* Thus the *Logarithms* of 4, 8, 16, 32, 64, &c. are found by *Multiplying* the *Logarithm* of the *Root* 2, by the *Indices* 2, 3, 4, 5, 6, &c. of those *Powers* respectively. Theref. if $x^n = a$, then n L, x = L, a.

So that an indifferent Capacity may hence see how the *Logarithms of Numbers* are made to any *exactness* or *Number of Places*; and therefore, how the whole *Table of Logarithms* may be Calculated anew, or how one already made may be *examined*, with all desirable *Facility* and *Dispatch*.

And as for the *Tables* themselves; 'tis not thought necessary here to insist on their various *Uses*, because they are so evident to them that understand what is already said, and so largely handled and exemplified by most Writers of *Practical Mathematics*.

COROLLARY.

Hence also, *from the Logarithm given, 'tis easy to find what Ratio it expresses.*

For L, $1 \pm x$ (or L) $= 1 \backsim \overline{1 \pm x}|^{\frac{1}{n}}$ ∴ $1 \pm \mathrm{L} = \overline{1 \pm x}|^{\frac{1}{n}}$

Conseq. $\overline{1 \pm \mathrm{L}}|^{n} = 1 \pm x$, that is, equal to

$1 \pm n\mathrm{L} + \frac{1}{2}n^2\mathrm{L}^2 \pm \frac{1}{6}n^3\mathrm{L}^3 + \frac{1}{24}n^4\mathrm{L}^4 \pm \frac{1}{120}n^5\mathrm{L}^5$, &c.

Therefore, if *Naper's Log.* be given (since $n = 1000$ &c.)

$1 \pm x = 1 \pm \mathrm{L} + \frac{1}{2}\mathrm{L}^2 \pm \frac{1}{6}\mathrm{L}^3 + \frac{1}{24}\mathrm{L}^4 \pm \frac{1}{124}\mathrm{L}^4$, &c.

Wherefore, if one of the *Terms* (a being the *Least*, and b the *Greatest*) of the *Ratio*, whereof L is the *Logarithm*, be given, the other is readily found; for in

Naper's Log. $\left\{ \begin{matrix} b = a \\ a = b \end{matrix} \right\} \times \overline{1 \pm \mathrm{L} + \frac{1}{2}\mathrm{L}^2 \pm \frac{1}{6}\mathrm{L}^3 + \frac{1}{24}\mathrm{L}^4, \text{\&c.}}$

Or

Or let the next neareſt *Logarithm* be call'd α, if *Leſs*, β if *Greater* than L, (its Num. a, or b) and $L-\alpha$, or $\beta - L = \delta$, then

$$\left.\begin{matrix} a \\ b \end{matrix}\right\} \times \overline{1 \pm \delta + \tfrac{1}{2}\delta^2 \pm \tfrac{1}{6}\delta^3 + \tfrac{1}{24}\delta^4 \pm \tfrac{1}{120}L^5, \&c.} = N,$$

the Number anſwering to the *Logarithm* L; where the *Series* converges according to the ſmallneſs of δ.

And the 1ſt. *Step* $a + a\delta$, $b - b\delta$, or $a + na\delta$, $b - nb\delta$, ſerves for the Common *Tables*.

Alſo $a + \dfrac{a\delta}{1-\frac{1}{2}\delta} - \dfrac{\frac{1}{24}a\delta^3}{1-\delta} + \dfrac{\frac{1}{35}a\delta^5}{1-2\delta}$ &c. $= N$.

Where the 1ſt. *Step* only, is ſufficient for Practice, in *Tables* exceeding any yet extant.

$$i.\ e.\ N = a + \frac{a\delta}{\frac{1}{n} - \frac{1}{2}\delta}, \text{ and } b - \frac{b\delta}{\frac{1}{n} + \frac{1}{2}\delta}.$$

$$\text{Or } \frac{\overline{\frac{1}{n} + \frac{1}{2}\delta} \times a}{\frac{1}{n} - \frac{1}{2}\delta}, \text{ and } \frac{\overline{\frac{1}{n} - \frac{1}{2}\delta} \times b}{\frac{1}{n} + \frac{1}{2}\delta} = N.$$

Theref. $\frac{1}{n} \mp \frac{1}{2}\delta : \frac{1}{n} \pm \frac{1}{2}\delta :: \left\{\begin{matrix} a \\ b \end{matrix}\right\} : N.$ ſought.

II. *To* Extract *the* Root *of an* Infinite Equation.

Suppoſe, for inſtance, the *Infinite Equation* to be $az + bz^2 + cz^3 + dz^4$, &c. $= \alpha y + \beta y^2 + \gamma y^3 + \delta y^4$, &c. Put $z = Ay + By^2 + Cy^3 + Dy^4 + Ey^5$, &c. Then by the 1ſt.

$az =$	$aA \times y$ +	$aB \times y^2$ +	$aC \times y^3$ +	$aD \times y^4$ +	$aE \times y^5$
$+bz^2 =$	:	$+bA^2$:	$+2bAB$:	$+bB^2$:	$+2bBC$:
	:	:	:	$+2bAC$:	$+2bAD$:
$+cz^3 =$	:	:	$+cA^3$:	$+3cA^2B$:	$+3cAB^2$:
	:	:	:	:	$+3cA^2C$:
$+dz^4 =$	:	:	:	$+dA^4$:	$+4dA^3B$:
$+ez^5 =$	:	:	:	:	$+eA^5$:
&c. $=$	$\alpha \times y$ +	$\beta \times y^2$ +	$\gamma \times y^3$ +	$\delta \times y^4$ +	$\epsilon \times y^5$ &c.

Where 'tis evident, by comparing the *Coefficients*, That

 A=

$$A = \frac{\alpha}{a}$$

$$B = \frac{\beta - bA^2}{a}$$

$$C = \frac{\gamma - 2bAB - cA^3}{a}$$

$$D = \frac{\delta - bB^2 - 2bAC - 3cA^2B - dA^4}{a}$$

$$E = \frac{\varepsilon - 2bBC - 2bAD - 3cAB^2 - 3cA^2C - 4dA^3B - eA^5}{a}$$

&c. Therefore, by *Substitution*,

$$z = \frac{\alpha}{a}y$$
$$+ \frac{\beta - bA^2}{a}y^2$$
$$+ \frac{\gamma - 2bAB - cA^3}{a}y^3$$
$$+ \frac{\delta - bB^2 - 2bAC - 3cA^2B - dA^4}{a}y^4$$
$$+ \frac{\varepsilon - 2bBC - 2bAD - 3cAB^2 - 3cA^2C - 4dA^3B - eA^5}{a}y^5$$

&c. Which is the *Theorem* given for this purpose, by that Ingenious Mathematician Mr. *De Moivre*, (in *Philos. Trans.* N° 240.)

And the *Observations* made for the *Continuance* of this *Series*, are hence also manifest; *viz.*

1. *That each Capital Letter is equal to the Coefficient of the Term preceding that where it was first express'd.*

2. *That the Denominator of each Coefficient must be always* a.

3. *That the 1st. Term of the Numerator must be a respective Coefficient in the Series,* $\alpha y + \beta y^2$, &c.

4. *That the Capital Letters must be Combin'd, as often as the Sum of their Exponents can be made equal to the Index of the Power to which they belong.*

5. *That there must be so many Capitals in each Term, as are denoted by the Exponents of the small Letter annex'd.*

6. *That the Capitals of every Member are capable of so many Permutations as are express'd by the Numeral Figures prefix'd.*

NOTE 1. The *Quantities* a, b, c, &c. α, β, γ, &c. are taken at Pleasure; Therefore may each represent an *Infinite Series* (if need be), or any *indetermined Quantities*; Consequently, if the *Equation* involve *more* than *two indetermined Quantities*, such also may easily be deduced from hence.

NOTE 2. This *Theorem* may be made Infinitely more *General*, by Substituting y^n, y^{2n}, y^{3n}, &c. in the room of y, y^2, y^3, &c. and putting *general Exponents* for *particular ones* in the other *Indeterminates*, then proceed as these *Exponents* require.

And those also may be made Infinitely more *General*, by observing the foregoing Method.

III. *To Investigate* General Ways *of* Extracting the Roots *of all sorts of* Equations.

Suppose any *Equation* whatsoever, as,

$x^n \cdot\cdot ax^{n-1} \cdot\cdot bx^{n-2} \cdot\cdot cx^{n-3} \cdot\cdot dx^{n-4}$, &c. $\cdot\cdot A = 0$

Whence A is the *Absolute known Quantity*;

x the *Root* required;

n the *Index* of the *Highest Power*;

1, a, b, c, &c. the respective *Coefficients*.

Put k (a *known Quantity*, taken at Pleasure, tho' the nearer the *True Root* the better) $\pm u$ (an *unknown Quantity*) equal to x; then the *Equation* will be

$$1\,\overline{k\pm u}|^{n-0} \ldots a\,\overline{k\pm u}|^{n-1} \ldots b\,\overline{k\pm u}|^{n-2}, \text{\&c.} \ldots A=0, \text{ or}$$

$$\pm 1\times k^{n-0}\ldots\overline{n-0}\,k^{n-1}\times u\ldots\frac{n-0}{1}\times\frac{n-1}{2}k^{n-2}\times uu, \text{\&c.}$$

$$\pm a\times k^{n-1}\ldots\overline{n-1}\,k^{n-2}\times u\ldots\frac{n-1}{1}\times\frac{n-2}{2}k^{n-3}\times uu, \text{\&c.}$$

$$\pm b\times k^{n-2}\ldots\overline{n-2}\,k^{n-3}\times u\ldots\frac{n-2}{1}\times\frac{n-3}{2}k^{n-4}\times uu, \text{\&c.}$$

$$\pm c\times k^{n-3}\ldots\overline{n-3}\,k^{n-4}\times u\ldots\frac{n-3}{1}\times\frac{n-4}{2}k^{n-5}\times uu, \text{\&c.}$$

$\pm$ &c.

$\mp A = 0$

$\underbrace{\quad}_{p} \;..\; \underbrace{\quad}_{q} \;..\; \underbrace{\quad}_{r}$ &c.

I. Now by rejecting the *Powers* of u, and their *Coefficients*, it may be express'd by a *Simple Equation*, and and its *True value* found by repeating the *Operation* so far as is necessary.

For since u is greater than it should be when it has a *Positive*, but less when it has a *Negative sign*; therefore 'tis plain, that if the *supposed Root* be *less* than the *required Root*, the following *Operation* will make it *greater*, and if it be *greater*, 'twill continue so, tho' *less* than the *former*, therefore the *Root* must necessarily *Converge*; and after an *Infinite Convergency* (if need be) must become equal to that sought: And therefore also the rest of the of the *Terms* may be safely rejected.

Theref.

Theref. $u = \frac{A \mp 1k^n \mp ak^{n-1} \mp bk^{n-2} \mp ck^{n-3}, \&c.}{nk^{n-1} \pm \overline{n-1}ak^{n-2} \pm \overline{n-2}bk^{n-3} \pm \overline{n-3}ck^{n-4}}$

And since $k + u = x$, by *Supposition*, therefore

$x = \frac{A \pm \overline{n-1}k^n \pm \overline{n-2}ak^{n-1} \pm \overline{n-3}bk^{n-2} \pm \overline{n-4}ck^{n-3}, \&c.}{nk^{n-1} \pm \overline{n-1}ak^{n-2} \pm \overline{n-2}bk^{n-3} \pm \overline{n-3}\,ck^{n-4}, \&c.}$

Whence *Particular Theorems*, (agreeing with those gi- by the Learned and Ingenious Mr. *Raphson*, in his *Analysis Equationum*) are easily drawn, for Extracting the Roots of all sorts of *Equations*, however *Compounded* or *Affected*.

And whatever *Term* is wanting in the *Equation*, must be omitted in the *Theorem*.

As 1. For all *Pure Powers*, *i. e.* if $x^n = A$,

Then $u = \frac{A - k^n}{nk^{n-1}}$; and $x = \frac{A + \overline{n-1}k^n}{nk^{n-1}}$

2. If $+ xx \pm ax = A$, then,

$u = \frac{A - k^2 \mp ak}{2k \pm a}$; and $x = \frac{A + k^2}{2k \pm a}$

And the like in any other *Equation*.

NOTE, That k may be taken at Pleasure; then in renewing the *Operation*, if you use the

First Theorem, Let $\left.\begin{matrix}1st.\\2d.\\3d.\\\&c.\end{matrix}\right\} k \pm \left\{\begin{matrix}1st.\\2d.\\3d.\\\&c.\end{matrix}\right\} u = \left\{\begin{matrix}2d.\\3d.\\4th.\\\&c.\end{matrix}\right\} k$: *Second Theorem*, Let $\left\{\begin{matrix}1st.\\2d.\\3d.\\\&c.\end{matrix}\right\} x = \left\{\begin{matrix}2d.\\3d.\\4th.\\\&c.\end{matrix}\right\} k$.

II. Since $x = k \pm u$, by *Supposition*; Therefore

$x^n = k^n \ldots \frac{n}{1}k^{n-1}u \ldots \frac{n}{1} \times \frac{n-1}{2}k^{n-2}uu \ldots \&c.$

And

And if x^n be $\left\{ \begin{matrix} \textit{greater} \\ \textit{Less} \end{matrix} \right\}$ than k^n, 'twill be $k \pm u$:

Let $(x^n \backsim k^n =) \frac{n}{1} k^{n-1} u .. \frac{n}{1} \times \frac{n-1}{2} k^{n-2} uu .. \&c. = m$

Therefore u taken equal to

$$m \div \left\{ \begin{array}{ll} \overline{\frac{n}{1} k^{n-1} (= a} & \textit{Doubles} \\ \overline{a \pm \frac{n}{1} \times \frac{n-1}{2} k^{n-2} u (= b)} & \textit{Triples} \\ \overline{b + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} k^{n-3} u^2 (= c)} & \textit{Quadr.} \\ \overline{c \pm \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} k^{n-4} u^3 (= d)} & \textit{Quint.} \\ \&c. & \&c. \end{array} \right\}$$

the *True Figures* in the assumed k, at each *Operation*.

Since, $u = \frac{m}{n k^{n-1}}$, or $\frac{m}{n k^{n-1} \pm \frac{1}{2} n \overline{n - n} k^{n-2} u}$

Therefore 'tis equal to

$$\frac{m}{n k^{n-1} \pm \frac{1}{2} \overline{n^2 - n} k^{n-2} \times \frac{m}{n k^{n-1}}} \text{ or } \frac{m}{n k^{n-1} \pm \frac{n-1}{2k} m}$$

That is $u = \frac{mk}{nk^n \pm \frac{1}{2} \overline{n-1}\, m}$, which is the *Rational Theorem* given by Mr. *Halley*, (in *Philos. Trans.* N. 210.) for *Extracting the Roots of all Pure Powers*.

Also

Also, $\pm nk^{n-1}u + \frac{1}{2}\overline{nn-n}k^{n-2}uu$, &c. $= m$, by *Hyp.*

Then $uu \pm \frac{2k}{n-1}u = \frac{m}{\frac{1}{2}\overline{nn-n}k^{n-2}}$ by *Equal Division.*

And $u = \mp\frac{k}{n-1} \pm \overline{\overline{\frac{k}{n-1}}\Big|^{\frac{2}{1}} \pm \frac{m}{\frac{1}{2}\overline{nn-n}k^{n-2}}}\Big|^{\frac{1}{2}}$, Theref.

$\overline{(k^n \pm m}\Big|^{\frac{1}{n}}$ or $k \pm u =) x = \frac{n-2}{n-1}k + \overline{\frac{1}{\overline{n-1}|^2}k^2 \pm \frac{m}{\frac{1}{2}\overline{nn-n}k^{n-2}}}\Big|^{\frac{1}{2}}$

Which is the *Irrational Theorem* given for the same purpose.

Note, If the *Root* of a very *high Power* be required; the renewing the *Theorem* will be somewhat troublesome: Therefore having found 3 or 4 *Figures* of u, the rest may be attain'd much easier, by applying the following *Correction*, viz. if it be $k \pm u$.

$$\frac{\frac{1}{2}\overline{n-2}x^3\frac{1}{k} \pm \frac{1}{3}\overline{n-2}\times\frac{1}{4}\overline{n-3}x^4\frac{1}{k^2} + \frac{1}{3}\overline{n-2}\times\frac{1}{4}\overline{n-3}\times\frac{1}{5}\overline{n-4}x^5\frac{1}{k^3}\ \&c.}{\overline{\frac{1}{\overline{n-1}|^2}k^2 \pm \frac{1}{\frac{1}{2}\overline{nn-n}k^{n-2}}m}\Big|^{\frac{1}{2}}\times 2}$$

In repeating the *Correction*, the u last found must be used; and in the 1st. 2d. 3d. &c. *Correction*, the *Dividend* must have 2, 4, 6, &c. *Terms*.

Also the *Divisor* must be corrected every *Operation*, by *Subducting*, or *Adding* the last *Correction*, *from*, or *to* it, if it be $k \pm u$.

Thus, if $x^{365} = 1{,}06$; 'tis found, with wonderful facility, by only one *Supposition* of $x = 1$, and *Corrections*, that $x = 1,\ 000159653587452947441 7154$, &c. which would have bin an intolerable Labour to perform by any other Method.

III. Whence also may be had variety of other *Theorems* for finding the *Roots* of *Equations*: By resuming the former *General Equation*, where

$$A = p \,..\, qu \,..\, ru^2 \,..\, su^3 \,..\, tu^4 \,..\, \text{\&c.}$$

And for *Equations* under the 9th *Dimension*, the following *Table* will abundantly assist the *Practitioner*; where $x^9 \,..\, ax^8 \,..\, bx^7 \,..\, cx^6 \,..\, dx^5 \,..\, ex^4 \,..\, fx^3 \,..\, gx^2 \,..\, hx \,..\, A = 0$.

	9	8	7	6	5	4	3	2	1	
p	k^9	ak^8	bk^7	ck^6	dk^5	ek^4	fk^3	gk^2	hk	A
q	$9k^8$	$8ak^7$	$7bk^6$	$6ck^5$	$5dk^4$	$4ek^3$	$3fk^2$	$2gk$	h	$\times u$
r	$36k^7$	$28ak^6$	$21bk^5$	$15ck^4$	$10dk^3$	$6ek^2$	$3fk$	g	$\times u^2$	
s	$84k^6$	$56ak^5$	$35bk^4$	$20ck^3$	$10dk^2$	$4ek$	f	$\times u^3$		
t	$126k^5$	$70ak^4$	$35bk^3$	$15ck^2$	$5dk$	e	$\times u^4$			
v	$126k^4$	$56ak^3$	$21bk^2$	$6ck$	d	$\times u^5$				
w	$84k^3$	$28ak^2$	$7bk$	c	$\times u^6$					
y	$36k^2$	$8ak$	b	$\times u^7$						
z	$9k$	a	$\times u^8$							

1. Observe, That the *Terms* in the *Ranks* p, r, t, &c. must have the *same Signs* with their respective *ones* in the *Equation*.

2. If it be $k \pm p$, then $k \mp u = x$.

3. If it be $k \pm u$, then the *Terms* in the *Ranks* q, s, v, must have the *same Signs with*, or *different from* their respective *ones* in the given *Equation*.

And since $p \,..\, qu \,..\, ru^2 \,..\, su^3 \,..\, tu^4$, &c. $.. A = 0$. Then $p = \,..\, qu \,..\, ru^2 \,..$ &c. Therefore u taken $=$ to

$p \div$	$q,$	*Doubles*	The *True Figures* in the assumed k, at each *Operation*.
	q, ru	*Triples*	
	q, ru, su^2	*Quadr.*	
	q, ru, su^2, tu^3	*Quint.*	
	&c.	&c.	

Since

Since $u = \frac{p}{q}$ or $\frac{p}{q \pm ru}$ or $\frac{p}{q \pm \frac{rp}{q}} = \frac{qp}{qq \pm rp}$

Theref. $x = k \pm \frac{qp}{qq \pm rp}$; which is the *Rational Theorem* for finding the *Roots* of *Affected Equations* of what *Power* soever.

Also because $qu \pm ru^2 = \pm p$ by *Supposition.*

Therefore, if p and A have *like*, or *unlike*; or p and r, *unlike*, or *like Signs*, 'twill be $k + u$, or $k - u$; and $u = \frac{\frac{1}{2}q \backsim \overline{\frac{1}{4}qq \pm pr}|^{\frac{1}{2}}}{r}$; $\therefore x = k \pm \frac{\frac{1}{2}q \backsim \overline{\frac{1}{4}qq \pm pr}|^{\frac{1}{2}}}{r}$ which is the *Irrational Theorem* for *solving Affected Equations.*

And by renewing the *Calculation* (if need be) and taking always the x *last found* for the k in the *New Equation*, 'tis easy to proceed to any exactness. But instead of renewing the *Theorem*, you may advantagiously use the following *Correction.*

viz. $\frac{su^3,\ tu^4,\ vu^5,\ wu^6,\ \&c.}{2 \times \overline{\frac{1}{4}qq \pm pr}|^{\frac{1}{2}}}$ or $\frac{\frac{1}{2}su^3,\ \frac{1}{2}tu^4,\ \frac{1}{2}vu^5\ \&c.}{\overline{\frac{1}{4}qq \pm pr}|^{\frac{1}{2}}}$ according as q and r have *like*, or *unlike Signs*. Where, in the *Dividend*, those *Terms* of the *Equation* that have the *same Signs with*, or *different from* s, must be $+$, or $-$.

And, in the *Divisor*, $\frac{1}{4}qq \pm pr$ *contrary to*, or *the same with* the *Sign* s, if it be $k - u$, but on the contrary, if it be $k + u$.

Also, in repeating the *Correction*, the *Divisor* may be corrected by *Subducting*, or *Adding* the last *Correction* multiplied by r, *from*, or *to* u, if s be *Affirmative*, or *Ne-*

gative; Then if q and r have the *same*, or *different Signs*, the *new Correction* must have the *same Signs with*, or *different Signs from* s.

EXAMPLE.

Suppose the *Value* of x in this *Cubic Equation*

$$x^3 + 438x^2 - 7825x - 98508430 = 0.$$
$$\text{or } xxx + axx - bx - A = 0. \text{ was required.}$$

Let $x = k \pm u$; (which k should be *assum'd* as near as possible to x, because convenient, but not necessary); therefore,

1. Suppose $k = 300$, then

$$\left.\begin{array}{lcl} x^3 & = & +27000000 + 270000u + 900u^2 + u^3 \\ +ax^2 & = & +39420000 + 262800u + 438u^2 \\ -bx & = & -2347500 - 7825u \\ -A & = & -98508430 \end{array}\right\} = 0$$

$$i.\,e. \left\{\begin{array}{l} -34435930 + 524975u + 1338u^2 = 0 \\ -\;p \;+\; qu \;+\; ruu = 0 \end{array}\right\} \therefore x = k + u$$

Then by the *Rational Theorem*

$$u\left(= \frac{p}{q + \frac{rp}{q}}\right) = 56,2;\ \text{theref. } x\,(= k + u) = 356,2-$$

And by the *Irrational Theorem*

$$u\left(= \frac{\overline{\tfrac{1}{4}qq + rp}|^{\frac{1}{2}} - \tfrac{1}{2}q}{r}\right) = 57;\ \text{theref. } x\,(= k + u) = 357+$$

2. Renew the Operation; And let $k = 356$, then,

$$\left.\begin{array}{lcl} x^3 & = & +45118016 + 380208u + 1068u^2 + u^3 \\ +ax^2 & = & +55510368 + 311856u + 438u^2 \\ -bx & = & -2785700 - 7825u \\ -A & = & -98508430 \end{array}\right\} = 0$$

$$i.\,e. \left\{\begin{array}{l} -665746 + 684239u + 1506u^2 = 0 \\ -\;p \;+\; qu \;+\; ruu = 0 \end{array}\right\} \therefore x = k + u$$

Then

Then by the *Rational Theorem*

$$u\ (=\frac{p}{q+\frac{rp}{q}})=,970894\ \therefore\ x=356,970894-.$$

And by the *Irrational Theorem*

$$u\ (=\frac{\overline{\frac{1}{4}qq+rp}|^{\frac{1}{2}}-q}{r}=,970898\ \because\ x=356,970898+$$

Or suppoſing $k=357$, Then,

$$\left.\begin{array}{llll} x^3 & =+45499293- & 382347u+ & 1071uu \\ +ax^2 & =+55822662- & 312732u+ & 438uu \\ -bx & \quad -2793525+ & 7825u & \\ -A & =-98508430 & & \end{array}\right\}=0.$$

$$i.e.\ \left\{\begin{array}{l} +20000-687254u+1509uu=0 \\ +\ p\ -\ qu\ +\ ruu\ =0 \end{array}\right\}\because x=k-u$$

Then by the *Rational Theorem*

$$u\ (=\frac{p}{q-\frac{rp}{q}})=,02910318169\because x=356,9708968183+$$

And by the *Irrational Theorem*

$$u\ (=\frac{\frac{1}{2}q-\overline{\frac{1}{4}qq-rp}|^{\frac{1}{2}}}{r})=,02910318180,\ \text{therefore},$$

$$x=356,9708968182-$$

If more Accuracy were required, repeating the *Operation*, or applying the *Correction*, would give any deſired Number of *Figures* true in the *Root*.

The vaſt Advantage of this *Method*, beyond any other whatever, will be evident to them that judiciouſly examine each; who will alſo find it, conſidering the ſeveral Compendiouſly eaſy ways of *Operation*, together with

with its Swiftness of *Converging*, so Compleat, that no *Useful Improvement* therein can be well expected.

CHAP. IX.

Of Geometric Progression.

DEFINITION.

A Continued Geometric Proportion, *that is, where the* Terms *do* Increase *or* Decrease *by* Equal Ratio's, *is called a* Geometric Progression; *thus,*

$$\left.\begin{array}{l} a,\ ar,\ arr,\ arrr,\ \&c.\ Incr. \\ a,\ \frac{a}{r},\ \frac{a}{rr},\ \frac{a}{rrr},\ \&c.\ Decr. \end{array}\right\}\text{from a } Continual \left\{\begin{array}{l} Mult. \\ Divis. \end{array}\right\}\text{by } r.$$

SCHOLIUM I.

But since this *Progression* is only a *Compound* of two *Series*, *viz.*

$$\text{Of}\left\{\begin{array}{l} Equals \\ Geom.\ Proport. \end{array}\begin{array}{llllll} a, & a, & a, & a, & a, & a, \\ 1, & r, & r^2, & r^3, & r^4, & r^5, \end{array}\right\}\&c.$$

Therefore, the most Natural *Progression* is that which begins with 1.

$$\left.\begin{array}{l} \text{As } \frac{1}{1}, \frac{r}{1}, \frac{r^2}{1}, \frac{r^3}{1}, \frac{r^4}{1}, \frac{r^5}{1} \\ i.e.\ 1,\ r,\ r^2,\ r^3,\ r^4,\ r^5 \end{array}\right\}\&c.\ Increasing.$$

$$\left.\begin{array}{l} \text{And } \frac{1}{1}, \frac{1}{r}, \frac{1}{r^2}, \frac{1}{r^3}, \frac{1}{r^4}, \frac{1}{r^5} \\ i.e.\ 1,\ r^{-1},\ r^{-2},\ r^{-3},\ r^{-4},\ r^{-5} \end{array}\right\}\&c.\ Decreasing.$$

SCHO-

SCHOLIUM 2.

In *Geometric Progression*;

If $\left\{\begin{matrix} a \\ r \\ n \\ l \\ s \end{matrix}\right\}$ be the $\left\{\begin{matrix} \text{First Term.} \\ \text{Ratio.} \\ \text{Number of Terms.} \\ \text{Last Term.} \\ \text{Sum of all the Terms.} \end{matrix}\right.$

Then any *three* of these *Terms* being *given*, the other *two* are easily found.

And the several *Cases* are reducible to *Ten Propositions*, which are all Solved by the *Two* following *Lemmata*.

I. Of *Increasing Progressions*.

LEMMA 1.

In an *Increasing Geometric Progression*.

$a,\ ar,\ ar^2,\ ar^3,\ ar^4,\ ar^5,\ ar^6,$ &c.

'Tis $1 : r :: s - l : s - a$.

For $a : ar :: s - l : s - a,$ \
But $a : ar :: 1 : r$ } by *Cor.* { 13, 5, 16 } *Th.* 2. \
Theref. $1 : r :: s - l : s - a,$

COROLLARIES.

1. $s = \dfrac{rl - a}{r - 1} = \dfrac{l - a}{r - 1} + l.$

2. $r = \dfrac{s - a}{s - l} = \overline{s - a} \times \dfrac{1}{s - l}$

3. $a = s + rl - rs = rl - s \times \overline{r - 1}.$

4. $l = \dfrac{rs - s + a}{r} = \dfrac{a + \overline{r - 1} \times s}{r} = s - \dfrac{s - a}{r}$

LEM-

LEMMA. 2.

In an *Increasing Geometric Progression.*

'Tis $1 : r^{n-1} :: a : l$.

For a, ar, ar^2, ar^3, ar^4, &c. $ar^{n-1} = l$
Theref. $1 : r^{n-1} :: a : l$.

CROLLARIES.

1. $l = ar^{n-1} = a \times r^{n-1}$
2. $a = \frac{l}{r^{n-1}} = l \times \frac{1}{r^{n-1}}$
3. $n = (\frac{L, r^{n-1}}{L, r} + 1 =) \frac{L, l - L, a}{L, r} + 1$
4. $r = \overline{l \div a}|^{\frac{1}{n-1}}$

PROPOSITIONS.

I. Given a, r, n; Required l, s.

1. $l = ar^{n-1}$ $(= a \times r^{n-1})$ by *Lem.* 2.

But $s = \frac{rl - a}{r - 1}$ by *Lem.* 1. And $r \times l = ar^n$ by *Mult.*

2. Theref. $s = \frac{ar^n - a}{r - 1}$ $(= a \times \frac{r^n - 1}{r - 1})$ by *Substitution.*

II. Given a, r, l; Required s, n.

1. $s = \frac{rl - a}{r - 1}$ $(= \frac{l - a}{r - 1} + l)$ by *Lem.* 1.
2. $n = \frac{L, l - L, a}{L, r} + 1$ by *Lem.* 2.

PRO-

III. Given a, r, s; Required l, n.

1. $= \frac{\overline{r-1}\times s + a}{r}$ $(= ar^{n-1})$ by *Lem.* 1*st.* and 2*d.*

Then $\overline{r-1}\times s + a\ (= r\times ar^{n-1}) = ar^{n}$ by *Mult.*

And $r^{n} = \frac{\overline{r-1}\times s + a}{a}$ by *Divis.* But nL,r = L,$\frac{\overline{r-1}\times s + a}{a}$,

2. Theref. $n = \frac{L,\overline{\overline{r-1}\times s + a} - L,a}{L,r}$ by *Division.*

IV. Given a, l, s; Required r, n.

1. $r = \frac{s-a}{s-l}$ $(= \overline{s-a} \times \frac{1}{s-l})$ by *Lem.* 1.

2. And $n\ (= \frac{L,l - L,a}{L,r} + 1) = \frac{L,l - L,a}{L,\overline{s-a} - L,\overline{s-l}} + 1$ (by *L.*2)

V. Given a, n, s; Required r, l.

Since $\frac{sr - s + a}{r}\ (= l) = ar^{n-1}$ by *Lem.* 1st. and 2d.

Then $sr - ar^{n} = s - a$ by *Divis.* and *Transp.*

1. Theref. $-r^{n} + \frac{s}{a}r = \frac{s-a}{a} = \frac{s}{a} - 1$, by *Divis.*

And since $l = ar^{n-1}$, and $r = \frac{s-a}{s-l}$, theref. $l = a \times \overline{\frac{s-a}{s-l}}\Big|^{n-1}$

2. Theref. $l \times \overline{s-l}|^{n-1} = a \times \overline{s-a}|^{n-1}$ by *Mult.*

VI. Given a, n, l, Required r, s.

1. $r = \overline{l \div a}|^{\frac{1}{n-1}}$ by *Lem.* 2. But $\frac{l-a}{r-1} + l = s$ by *L.* 1

2. Theref. $s = \frac{l-a}{\overline{l \div a}|^{\frac{1}{n-1}} - 1} + l$, by *Substitution.*

VII. Given r, n, l; Required a, s.

1. $a = \frac{l}{r^{n-1}}$ by *Lem.* 2. But $\frac{lr-a}{r-1} = s$, by *Lem.* 1.

2. Theref. $s = \frac{lr - \frac{l}{r^{n-1}}}{r-1} = \frac{lr^n - l}{r^n - r^{n-1}}$ by *Substitution.*

VIII. Given r, n, s; Required a, l.

Since $sr - s + a = ar^n$ by *Lem.* 1st. and 2d.

Then $sr - s = ar^n - a$ $(= a \times \overline{r^n - 1})$ by *Transp.*

1. Theref. $a = \frac{sr-s}{r^n-1} = \frac{r-1}{r^n-1} \times s$ by *Division.*

And since $s = \frac{lr^n - l}{r^n - r^{n-1}}$ by *Prop.* 7. ∴ $sr^n - sr^{n-1} = lr^n - l$;

2. Theref. $l = \frac{sr^n - sr^{n-1}}{r^n - 1}$ by *Division.*

IX. Given r, l, s; Required a, n.

1. $a = s + rl - rs$ $(= lr - s \times \overline{r-1})$ by *Lem.* 1.

But $\frac{l}{a}$ $(= \frac{l}{s + rl - rs}) = r^{n-1}$ by *Lem.* 2.

And

And L, $\frac{l}{s+rl-rs} = \overline{n-1}$ L, r by the *Nat.* of *Log.*

2. Theref. $n = \frac{L,l - L,\overline{s+rl-rs}}{L,r} + 1$, by *Div.* and *Tr.*

X. Given n, l, s; Required r, a.

Since $sr^n - sr^{n-1} = lr^n - l$, by *Prop.* 8.

Then $l = sr^{n-1} - sr^n + lr^n$ $(= sr^{n-1} - \overline{s+l}\times r^n)$ by *Transf.*

1. Theref. $-r^n + \frac{s}{s-l}r^{n-1} \left(= \frac{s}{s-l}r^{n-1} - r^n\right) = \frac{l}{s-l}$

2. $a \times \overline{s-a}|^{n-1} = l \times \overline{s-l}|^{n-1}$ by *Prop.* 5.

II. *Of Decreasing Geometric Progressions.*

In *Finite Decreasing Progressions*, the same *Rules* will serve for the like *Propositions*, if the *Series* be inverted, so that the *Least Term* be the *First*, and the *Greatest* the *Last*.

And since in the *Increasing Geometric Progression*

$a, ar, ar^2, ar^3, ar^4, ar^5$, &c. to $ar^{n-1} = l$.

'Tis $r - 1 : 1 :: l - a : s - l$.

Therefore in a *Decreasing Geometric Progression*

'Tis $r - 1 : 1 : a - l : s - a$. by Inverting the *Terms*

COROLLARIES.

1. But in an Infinite Decreasing Progression $l = 0$; Therefore $r - 1 : 1 :: a : s - a$. whence,

Prop.	Given	Requd.	Solution
1.	, a.	s.	$r\,a \div \overline{r - 1}$
2.	s, a.	r.	$s \div \overline{s - a}$
4.	s, r.	a.	$s - \overline{s \div r}$

2. Also $a - \frac{a}{r} : \frac{a}{r}$ (*i.e.* 1*st.*—2*d.* : 2*d.* or x) :: $r - 1 : 1$.

Theref. $a - x : x :: \left\{ \begin{matrix} a - l \\ a \end{matrix} \right\} : s - a$, in $\left\{ \begin{matrix} \text{a } \textit{Finite} \\ \text{an } \textit{Infinite} \end{matrix} \right\}$ *Progr.*

And $s = \frac{a\,a - x\,l}{a - x}$ in a *Finite*, or $s = \frac{a\,a}{a - x}$ in an *Infinite Decreasing Progression.*

Question. *Suppose a Body should move at this rate*, viz. *in the 1st. Moment* 10 *Miles, in the 2d.* 9 *Miles, in the 3d.* $8\frac{1}{10}$ *&c. eternally, as* 10 *to* 9;

Here is given $r = \frac{10}{9}$, $a = 10$; required s; then,

By *Cor.* $\left\{ \begin{matrix} 1 \\ 2 \end{matrix} \right\} s = \left\{ \begin{matrix} r\,a \div \overline{r - 1} \\ a\,a \div \overline{a - x} \end{matrix} \right\} = 100$ *Miles* sought.

That is, a *Moveable Body* continuing its Motion in that Ratio *eternally*, would only run 100 *Miles*, or more than any thing that is less than 100 *Miles*.

3. Since $r - 1 : 1 :: a : s - a$, therefore,

$$s - a = \frac{a}{r - 1} = \overline{a + \frac{a}{r} + \frac{a}{r^2} + \frac{a}{r^3}}, \text{\&c.} - a = \frac{a}{r} + \frac{a}{r^2} + \frac{a}{r^3} \text{ \&c.}$$

Whence, *if any Quantity* a *be continually divided by any other Quantity* r, *the Sum of all the Terms will be* $\frac{a}{r - 1}$

i. e

i. e. $\frac{1}{r-1} \times a$ or the $\frac{1}{r-1}$ of a.

Theref. $a \times \frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3} + \frac{1}{r^4}$, &c. $= a \times \frac{1}{r-1}$

Or, $\frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3} + \frac{1}{r^4}$, &c. $= \frac{1}{r-1}$

Where, if $a = r - 1$, then $s = 1$.

4. Whence, 'tis evident, That an *Infinite Progreffion*, or an *Infinitely Infinite one*, may be Collected into one *Sum*; which *Sum* may not be only *Finite*, but equal to *Nothing*.

And of *Infinites* 'tis hence plain, that fome are *equal*, others *unequal*; and alfo that one *Infinite* may be equal to *Two* or *more Finites*, or *Infinites*.

CHAP. X.

Of Intereft.

I. *Of a* Single Sum *of Money* Paid *either* Before, *or* After *'tis* Due.

1. Allowing Simple Interest.

Put p = *Principal*, or *Sum forborn*.
n = *Number of Years*, or *parts of a Year*.
r = *Rate of* 1 l. per Annum.
m = *Amount* of the faid *Principal*, for that *Time*, at that *Rate*.

Since

Since the *Amount* of 1 *l.* for 1 *Year*, is $1+r$;

Therefore the *Amount* of 1 *l.* for *n* Years is $1+nr$;

And 1 *l.* : $1+nr :: p : p+pnr = m$, the *Amount*, and $prn =$ *Interest* of the *Principal* p, at the *Rate* r, in *n Years*. Hence,

Probl.	Given	Reqd.	~~Given~~ Solution
1.	$p, r, n.$	$m.$	$p \times \overline{1+rn}$
2.	$m, r, n.$	$p.$	$m \div \overline{1+rn}$
3.	$m, p, r.$	$n.$	$\overline{m-p} \div pr$
4.	$m, p, n,$	$r.$	$\overline{m-p} \div pn$

Whence variety of easy *Rules* for *Practice* may be deduced; such as, *To find the Interest of any Sum, at any Rate, for any Number of days.*

'Tis plain, that $\overline{n \text{ Days} \times p \text{ Pence}} \div 7300$ (= 365 $\frac{100}{5}$) is the *Interest* in *Pence*, at 5 *l. per Cent. per An.* Therefore,

$\frac{2r}{10} \times \frac{np}{7300}$, or $2r \times \frac{np}{73} \div 1000$, or $2r \times \frac{np}{7300} \div 10$. is the *Interest* in *Pence*, at the *Rate* r.

2. Allowing Compound Interest.

Put $x\ (=1+r) =$ *Principal* and *Interest* of 1 *l.* for any given *Time*, at any given *Rate*. Then,

Since, 1*l.* : $x :: p : px = $ 1st.
1*l.* : $x :: px : px^2 = $ 2d.
1*l.* : $x :: px^2 : px^3 = $ 3d.
&c. &c.
} Year *Amount* (m).

Therefore 'tis $px^n = m$; whence,

Prob.

Probl.	Given	Reqd.	Solution
1.	$p, n, x.$	$m.$	$p \times x^n$ or $L, p + \overline{L, x} \times n$
2.	$m, n, x.$	$p.$	$m \div x^n$ or $L, m - \overline{L, x} \times n$
3.	$p, m, x.$	$n.$	$x^n \div\div x$ or $L, x^n \div L, x$
4.	$p, m, n.$	$x.$	$\overline{m \div p}\vert^{\frac{1}{n}}$ or $L, m - \overline{L, p} \div n$

Queſt. 1. *Suppoſe the Principal* p *were at Intereſt for* n *Times at the Rate* r; *what is the Amount* m?

Queſt. 2. *If the Sum* m *was to be paid* n *Times hence*; *what is its preſent Value* p *diſcounting at the Rate* r?

Queſt. 3. *If the Principal* p *at the Rate* r *gives the Amount* m; *what is the Time of forbearance* n?

Queſt. 4. *If the Principal* p *forborn* n *Times gives the Amount* m; *what is the Rate* r *of Intereſt*?

II. *Of ſeveral* Equal Payments *at ſeveral* Equal Times.

Firſt, *When paid after they are due*;

Or *Rules* for finding the *Amount*, &c. of *Annuities*, *Penſions*, or *Rents*, *&c.* in *Arrear*.

1. Allowing Simple Interest.

Put a for the *Annuity*, *Rent*, or *Penſion*; then,
'Tis evident, 1 l. : r :: a l. : ar, that is,
The *Intereſt* of a l. at the *Rate* r, *per* 1 l. *per An.* is ar.

$$\text{But} \left\{\begin{array}{l} a \\ a + 1ar \\ a + 2ar \\ a + \overline{n-1} \times ar \end{array}\right\} \text{is the} \left\{\begin{array}{l} \text{1ſt.} \\ \text{2d.} \\ \text{3d.} \\ n\text{th.} \end{array}\right\} \text{Years } Amount \text{ (m).}$$

Therefore

Therefore $na + \frac{nn - n}{2} ar$ is the *Sum* of those *Amounts*, or the *Amount* (m) of the *Annuity* (a) at the *n*th. *Year*'s end; whence,

Probl.	Given	Requd.	Solution
1.	$a, n, r;$	$m.$	$na + \frac{nn - n}{2} \times ar$
2.	$m, n, r;$	$a.$	$\frac{2m}{2 + nr - r \times n}$
3.	$m, n, a;$	$r.$	$\frac{m - na \times 2}{n - 1 \times na}$
4.	$m, r, a;$ Let $2a - ra = z$	$n.$	$\frac{z + \overline{zz + 8mra}\vert^{\frac{1}{2}}}{2ra}$

2. Allowing Compound Interest.

Since the *Last* Year's *Annuity* (a) tarries out no *Time*, therefore no *Interest* can be demanded for it; consequently the *First* Year's *Annuity* will become ax^{n-1},

Wherefore, $a + ax + ax^2 + ax^3 + ax^4$, &c. to ax^{n-1} $= m$ the *Amount*

But $a : ax$ \
That is $1 : x$ } $:: m - ax^{n-1} : m - a$, by *Cor.* 13. *Th.* 2.

Therefore $mx - ax^n = m - a$, whence;

Probl.

Probl.	Given	Reqd.	Solution
1.	*a, x, n.*	*m.*	$\frac{\overline{x^n-1}\times a}{x-1}$
2.	*m, x, n.*	*a.*	$\frac{\overline{x-1}\times m}{x^n-1}$
3.	*m, x, a.*	*n.*	$\frac{L,\overline{\overline{x-1}\times m+a}-L,a}{L,x}$
4.	*m, a, n.*	*x.*	$-x^n+\frac{m}{a}x=\frac{m-a}{a}$

Queft. 1. *If the Annuity* a *be forborn* n *Times; at the the rate* r; *what Amount* m *will it arife to?*

Queft. 2. *If in* n *Times forbearance, at the Rate* r *the Amount* m *is raifed; what was the Annuity* a *forborn?*

Queft. 3. *If the Annuity* a, *at the Rate* r, *raife or amount to* m; *what was the Time* n *of forbearance?*

Queft. 4. *If the Annuity* a *forborn* n *Times, raife the Stock or Amount* m; *what was the Rate* r *of Intereft?*

Secondly, *Being paid before they are due.*

Or *Rules for finding the Difcount,* &c. in *Buying* and *Selling* of *Annuities, Penfions* and *Leafes* in *Reverfions,* &c.

1. Allowing Simple Interest.

Since, *As the Amount of* 1 l. *for any Time is to* 1 l. *So is the Amount of an Annuity, to its Prefent value.*

i. e. $1+nr : 1 :: na+\frac{nn-n}{2}ar : \frac{\overline{na+\frac{1}{2}\overline{nn-n}\times ar}}{1+nr}=s.$

E e Therefore

Therefore,

Probl.	Given	Reqd.	Solution
1.	$a, r, n;$	$s.$	$\dfrac{\overline{2n+nnr-nr}\times a}{2+2nr}$
2.	$s, r, n;$	$a.$	$\dfrac{\overline{2+2nr}\times s}{\overline{2+nr-r}\times n}$
3.	$s, n, a;$	$r.$	$\dfrac{\overline{na-s}\times 2}{\overline{2s-an+a}\times n}$
4.	$s, r, a;$ Let $2sr+ra-2a=z$	$n.$	$\dfrac{z+\overline{zz+8sar}\vert^{\frac{1}{2}}}{2ra}$

2. Allowing Compound Interest.

Since,

$$\left.\begin{array}{l} x : 1\,l. :: a : \dfrac{a}{x} \\ x : 1\,l. :: \dfrac{a}{x} : \dfrac{a}{x^2} \\ x : 1\,l. :: \dfrac{a}{x^2} : \dfrac{a}{x^3} \\ \text{\&c.} \end{array}\right\} = \text{Present Value } (s) \text{ at the } \left\{\begin{array}{l} \text{1st.} \\ \text{2d.} \\ \text{3d.} \\ \text{\&c.} \end{array}\right\} \textit{Year's end.}$$

Theref. $\dfrac{a}{x}+\dfrac{a}{x^2}+\dfrac{a}{x^3}+\dfrac{a}{x^4}$, &c. to $\dfrac{a}{x^n}$ = *Present value* s.

But $\left.\begin{array}{l}\dfrac{a}{x} : \dfrac{a}{x^2} \\ \text{or } x : 1\end{array}\right\} :: s-\dfrac{a}{x^n} : s-\dfrac{a}{x}$ by *Cor.* 13. *Th.* 2. *Ch.* 3.

Theref. $s-\dfrac{a}{x^n}=sx-a$: Whence follows,

Probl.

Probl.	Given	Reqd.	Solution
1.	$a, x, n.$	$s.$	$\overline{a - \frac{a}{x^n}} \div \overline{x - 1}$
2.	$s, x, n.$	$a.$	$\frac{x^n \times \overline{x - 1} \times s}{x^n - 1}$
3.	$s, x, a,$	$n.$	$\frac{L,a - L,\overline{a + s - sx}}{L,x}$
4.	$s, a, n,$	$x.$	$-x^{n+1} + \frac{a+s}{s} x^n = \frac{a}{s}$

Queft. 1. *The Annuity* a *is to be ſold for* n *Years, allowing the Purchaſer the Rate* r; *what is the Preſent worth* s *of the Annuity?*

Queft. 2. *Having the Sum* s *ready to be laid out, at the Rate* r, *to buy an Annuity for* n *Years; what Annuity will it Purchaſe?*

Queft. 3. *The Annutity* a *is to be ſold for* n *Years, for the ready Sum* s; *what rate* r *has the Purchaſer for his Money?*

Queft. 4. *The Annuity* a *is made over for Payment of a Debt* s, *allowing the Creditor the Rate of Intereſt* r; *In what Time* n *will the Debt be paid?*

Several other more Practical Rules for ſolving the laſt *Problem* in this, and the former *Caſe*, are eaſily found; we as ſhall ſhew hereafter.

COROLLARY 1.

By Suppoſing n, in the laſt *Theorems*, to be Infinite, and a the *Annual Rent*; it follows that $s = sx - a$.

Whence, *Rules* are drawn for *Buying* and *Selling* of *Eſtates* in *Fee-Simple*, allowing *Com. Intereſt*.

Ee 2 Prol.

Prop.	Given	Reqd.	Solution
1.	$a, x.$	$s.$	$a \div \overline{x-1}$
2.	$s, x.$	$a.$	$s \times \overline{x-1}$
3.	$s, a.$	$x.$	$\overline{s+a} \div s$

Queſt. 1. *There is a Fee-Simple to be ſold, of* a *l.* per Annum, *allowing the Purchaſer the Rate* r, *Comp. Intereſt; what Sum* s *will Purchaſe this Eſtate?*

Queſt. 2. *There is a Sum* s, *ready to be laid out, at the Rate* r, *Comp. Intereſt, for Buying a Fee-Simple; what Yearly Rent can ſuch an Eſtate be of?*

Queſt. 3. *Having with the Sum* s *bought a Fee-Simple, whoſe Yearly Value is* a *l. what Rate* r *of Comp. Intereſt was allow'd for the Money?*

COROLLARY 2.

Whence alſo, If it be required, *how many Years Purchaſe, any Annuity is worth, allowing Comp. Intr.*

Suppoſe N the *Number of Years* ſought;
Now, that $Na = s$ is evident, and $a = sx - s$ by *Prec.*

Theref. $N \left(= \frac{s}{sx - s}\right) = \frac{1}{x - 1}$ by *Diviſion*; hence,

Probl.	Given	Reqd.	Solution
1.	$x.$	N.	$1 \div \overline{x-1}$
2.	N.	$x.$	$\overline{N+1} \div N$

Queſt. 1. *There is a Fee-Simple to be ſold; how many Years Purchaſe* N *is it worth, allowing the Purchaſer the Rate* r *Comp. Intereſt?*

Queſt. 2. *There is a Fee-Simple bought, and* N *Years Purchaſe given for it; what Rate* r *of Com. Intereſt is the Money valued at?*

CHAP. XI.

I. *Of* Combinations *of* Quantities.

DEFINITION.

The ſeveral Ways *or Different* Caſes, *of* Taking *or* Leaving *any Number of* Quantities, *out of any Number of* Things *expoſed, without regarding their* Order *or* Places, *are called the* Combinations *of* Quantities.

Let N be the *Number* of *Things* expoſed.

n the *Number* of *Quantities* to be *taken* or *left*.

1. The *Combinations* of 2 *Quantities*, in theſe 6, *viz.* *a*, *b*, *c*, *d*, *e*, *f*, are 15.

For with
- *a* —— *ab*, *ac*, *ad*, *ae*, *af*; 5.
- *b* ———— *bc*, *bd*, *be*, *bf*; 4.
- *c* ———— —— *cd*, *ce*, *cf*; 3.
- *d* ———— ———— *de*, *df*; 2.
- *e* ———— ———— ———— *ef*; 1.

Therefore the *Combinations* of 2 in 6 is 15; which is a *Figurate Number* of the 3d. *Order*, whoſe *Side* is is N—1, or $N - \overline{n - 1}$; Because 'tis the *Aggregate* of a *Series* of the 2d. *Order*.

2. The *Combinations* of 3 *Quantities* in these 6, are 20, *viz.*

with *a*,	with *b*,	with *c*,	with *d*.
abc, *abd*, *abe*, *abf*; 4.	*bcd*, *bce*, *bcf*; 3.	*cde*, *cdf*; 2.	*def*; 1
acd, *ace*, *acf*; 3.	*bde*, *bdf*; 2.	*cef*; 1.	1
ade, *adf*; 2.	*bef*; 1.	3	
aef; 1	6		
10			

And $10 + 6 + 3 + 1 = 20$; which is a *Figurate Number* of the 4th *Order*, whose side is N — 2, or $N - \overline{n - 1}$; Because 'tis the *Aggregate* of a *Series* of the 3d *Order*: The same in any other.

Hence, $n + 1$ expresses what *Order* of *Figurate Numbers* to take.

$$\text{If } n = \left\{\begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ \&c. \end{matrix}\right\} \text{ then } \left\{\begin{matrix} 1 = s, \text{ or Side of the Fig. Number.} \\ N - 0 \\ N - 1 \\ N - 2 \\ \&c. \end{matrix}\right\} = s = N - \overline{n - 1}.$$

Consequently, $\frac{s+0}{1} \times \frac{s+1}{2} \times \frac{s+2}{3}$, &c. to $\frac{s + \overline{n-1}}{n}$ shall be the *Figurate Number* required, or the Number of *Combinations* of n *Quantities* in N. by *Prob.* 4. *Ch.* 8.

SCHOLIUM.

Whence, the usual *Examples*, given by *Writers* on this Subject, are easily *Solv'd*; and the Reason of those *Solutions* made evident.

II. Of

II. *Of* Elections *of* Quantities.

The Sum *of all the* Combinations *found by taking* 0, 1, 2, 3, &c. Quantities *out of any Number of* Things *exposed, is called the* Election *of* Quantities.

Thus in 6 *Quantities*, the *Elections* are 64.

$$\text{The } \textit{Combinations} \text{ of} \left\{\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}\right\} \text{ in } 6, \text{ is } \left\{\begin{array}{c} 1 \\ 6 \\ 15 \\ 20 \\ 15 \\ 6 \\ 1 \\ \hline \end{array}\right.$$

Therefore the *Election* of 6 *Quantities* is 64: And so in any other *Number*.

Hence 'tis evident, that the several *Combinations* in any Number, are only the *Unciæ* of the several *Terms* of a *Binomial Root*, rais'd to the Power N; or the Members of the respective *Powers* of $1+1$ taken as a *Binomial*:

Therefore $\overline{1+1}|^{N}$ or 2^{N} is the *Election* of N *Quantities*, or the Number of Varieties in *taking* or *leaving*, 0, 1, 2, 3, &c. of any *Number of Things* expos'd, without regarding their *Order*.

Note, If the *taking* of 0 be excluded; *Then* $2^{N}-1$ *will be Election of N Quantities*.

But if the *taking* of 0 and 1 be excluded; *Then* $2^{N}-\overline{N+1}$ *will be the Elections of N Quantities*.

III Of

III. *Of* Permutations *of* Quantities.

DEFINITION.

The several ways, that the Order *of a certain Number of* Different Quantities, *may be* Varied *or* differently Placed, *are called* Alterations, Variations, *or* Permutations *of* Quantities.

As suppose the *Quantities* were *a*, *b*, *c*, *d*, &c. 'Tis plain,
In 1 *Quant. a.* In 2 *Quant. a*, *b*. In 3 *Quant. a*, *b*, *c*, &c.

$$\{a\} \qquad 1 = 1 \text{ var.}$$

$$\left\{\begin{matrix} a\,b \\ b\,a \end{matrix}\right\} \qquad 1 \times 2 = 2 \text{ variat.}$$

$$\left\{\begin{matrix} a\,b\,c \\ a\,c\,b \\ b\,a\,c \\ b\,c\,a \\ c\,a\,b \\ c\,b\,a \end{matrix}\right\} \qquad 1 \times 2 \times 3 = 6 \text{ Var.}$$

Theref. $N \times \overline{N-1} \times \overline{N-2} \times \overline{N-3}$, &c. to $\overline{N-N}$ is the *Number of Alterations*, or *Changes of Order in* N *Quantities*

1. *Hence the Changes that any Number of Bells admit of, are readily found.*

2. *As also, how many ways, the Letters of a Name, or Word may be differently disposed of, by way of Annagram*; As the word *Roma* admits of $4 \times 3 \times 2 \times 1 = 24$ *Changes*.

But if one, or more of the *Letters*, do occur more than once; *Then the whole Number of Changes, must be divided by the Product of the Changes of the Repetition of those Letters*; Thus,

a a a b c admits of $\frac{5 \times 4 \times 3 \times 2 \times 1}{6 \times 2} = \frac{120}{12} = 10$ *Changes*.

IV. Com-

IV. Composition of Quantities.

DEFINITION.

The several Dispositions *of any Number of* Quantities *placed according to all the* different *ways, they are capable of, by taking them either singly, or connecting the given* Rank *to each* Term *therein, as often as required, are call'd* Composition of Quantities: *Thus,*

The *Composition* of 2 *Quantities* in these 3, *a, b, c,* is 9.

For with *a. b. c.*

a a	*a b*	*a c*
b a	*b b*	*b c*
c a	*c b*	*c c*

$3 + 3 + 3 = 9$

And the *Composition* of 3 *Quantities* in *Three* is 27. or *Generally* N^n is the *Composition* of *n Quantities* in N.

COROLLARY.

Whence because N^1, N^2, N^3, N^4, &c. to N^n or N^N is a *Geometric Progression*, whose *First Term* $=$ N $=$ *Ratio*, and *Number* of *Terms* $= n$. Therefore

its *Sum* $(N \times \frac{N^n - 1}{N - 1} = \frac{NN^n - N}{N - 1} =) N^n - 1 + \frac{N^n - 1}{N - 1}$

(by *Prop.* 1. *Ch.* 9.) will give the several *Dispositions* there may be of N *Quantities*, taking them *one* and *one*, *two* and *two*, &c. to N *Quantities*.

SCHOLIUM.

And since 24^{24} $(N^N) = 1333735776850284124449$ 081472843776 is the Number of the several *Compositions* that 24 *Letters* make, by taking each time 24; therefore, $(N^N - 1 + \overline{\overline{N^N - 1} \div \overline{N - 1}} =)$ 1391724288887292 999425128493402200, is *the Number of all the words significant and insignificant, that can be made of 24 Letters.*

Variety of *Instances* might have bin given on the several Heads of this *First Part*, but because the Application is endless, the Learner (in most Places) is left to his Liberty of taking what Examples he pleases: For we would use as much brevity as possible, in Order that other useful things design'd for this Treatise, may come in: And 'tis hoped what has already bin laid down, if well regarded, is sufficient without any farther illustration, to render the whole *Art of Numbering*, whether *Theorical* or *Practical*, intelligible to any reasonable Capacity.

The *Symbols* used in the following Part, are only the Common ones, as $\angle$, for *Angle*; $\llcorner$, for *Right Angle*; $\perp$ for *Perpendicular*; $\parallel$, for *Parallel*; ∞, for *Infinite*: with other *Abbreviations* of words, which the Reader will easily understand.

SYNOP-

SYNOPSIS *Palmariorum Matheseos.*

PART II.

Containing the PRINCIPLES OF GEOMETRY.

1. THAT Science by which we learn how to *Compare* Things extended the one with the other, or determine how the one is More Less, or as Much extended as another, is called *GEOMETRY.*

2. In Things extended we diftinguifh three *Dimenfions*, viz. *Length*, *Breadth*, and *Thicknefs*.

The Quantity wherein we conſider	*Length, Breadth* & *Thickneſs*	is call'd	*Solid.*
	only *Length*, and *Breadth*		*Surface.*
	nothing but *Length*		*Line.*

And that wherein we conſider neither *Length, Breadth*, or *Thickneſs*, and therefore is conceiv'd to have no Parts, or whoſe Diviſibility is not conſider'd, is called a *Point.*

3. *Lines*, *Surfaces*, and *Solids* may be conſider'd as generated by a continual Motion of *Points*, *Lines*, and *Surfaces:* And as theſe Motions may be different, ſo may there be different Sorts of *Lines*, *Surfaces*, and *Solids* deſcribed by them.

4. If a *Point* move towards another the neareſt way, 'twill deſcribe a *Right Line*, otherwiſe a *Curv'd* one: So a *Right Line* moved on a *Right*, or *Curved Line*, will produce a *Plane*, or *Curved Surface*: The ſame may be conceived of *Solids*.

5. The Curve-line whoſe parts are equally diſtant from a Point in the ſame Plane, is called a *Circumference*; that Point the *Centre*; Right Lines drawn from the Centre to the Circumference are called *Radii:* Thoſe drawn from one Point of the Circumference to the other are called *Chords*; and that Chord which paſſes thro' the Centre is called a *Diameter* and divides the Circumference equally in two. Alſo any *Circumference* may be conceived to be divided into 360 equal parts called *Degrees*, and each Degree into 60 equal parts, called *Minutes*, and each of thoſe again ſubdivided into 60 *Seconds*, &c. Any part of the Circumference is called an *Arc.*

6. The Inclination of two Right Lines meeting in a Point, ſo as not to make one Right Line, is called a *Plane Angle*; which therefore may be conceived to be generated by the Rotation of Sides: And its *Meaſure* is an Arc deſcribed from the Angular Point as a Centre, and intercepted between the Lines which form it. An *Angle* is ſaid to be *Equal* to, *Greater*, or *Leſs* than another, according as the Arc which meaſures it contains, as *Many*, *More*, or *Fewer* of the equal parts into which that Circumference is ſuppoſed to be divided.

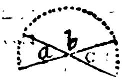
7. When

7. When a Right Line ſtands upon another, ſo as to make equal *Inclinations* or *Angles* on each Side thereof, that Line is called a *Perpendicular*; and is the neareſt Diſtance between a given Point and a given Line; thoſe equal *Angles* are called *Right Angles*; which therefore have each for their Meaſure $\frac{1}{4}$ of the *Circumference* or 90 *Deg.* Conſeq. $\frac{1}{2}$ the *Circumf.* = two *Right Angles*; and the whole *Circumf.* = 4 ∟s. An Angle *greater*, or *leſs* than a *Right Angle* is called *Obtuſe*, or *Acute*.

8. Half the Chord of twice any *Arc* is called the *Right Sine* of that *Arc*: That part intercepted between the *Arc* and its *Right Sine*, is called the *Verſed Sine* thereof. And the *Sine* of an Angle is the *Sine* of the *Arc* which meaſures that Angle; and is a Perpendicular from one end of the Arc, on a Right Line drawn from the Angular Point to the other.

9. *A Right Line falling upon another makes two Angles equal to two Right Angles.* For they are meaſured by $\frac{1}{2}$ the Circumference, which meaſures 2 ∟s (by 7.) *Conſequently all the* ∠s *made at the ſame Point, on the ſame ſide of a Right Line, are* = 2 ∟s. *Therefore, that is a Right Line on which another falling makes two* ∠s = to 2 ∟s. *And all the* ∠s *that can be made about any Point are* = 4 ∟s.

10. *Therefore the op.* ∠s. (a, c) *of croſſing Lines are* =; For $a + b = 2$ ∟s $= b + c$ (by 9) therefore $a = c$ (by *Subduction.*)



11. *A Tangent to any Point of the Circumference is Perpendicular to the Radius drawn to that Point.* For the Radius is the neareſt Diſtance.

12. A Right Line drawn *Perpendicular* to the Radius, paſſing thro' one end of an Arc, and limited by a Line, called *Secant*, drawn from the Centre thro' the other end, is the *Tangent* of that Arc.

13. The *Radius* being ſuppoſed divided into any Number of = parts; the Quantity of the *Sines, Tangents,* &c. of all *Angles* are eſtimated according as they contain more or fewer of ſuch Parts. Therefore,

In

In the same or equal Circumferences, the Sines, Tangents, &c. of equal Arcs or Angles are respectively equal. Because they consist of an equal Number of Parts of the same or equal *Radii.*

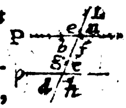
In unequal Circumferences, the Sines, Tangents, &c. of equal ∠ *s, or Similar Arcs are Similar also.* Because they contain the like Number of Parts, of their respective *Radii.*

14. The Difference of an Arc from ½, or ¼ the Circumference is called the *Supplement*, or *Complement* of that *Arc*. An *Arc* and its *Supplement*, have the same *Sine, Tangent*, and *Secant*. And the *Sine, Tangent*, or *Secant* of the *Complement*, is called the *Co-Sine, Co-Tangent*, or *Co-Secant* of the *Arc*, whose *Complement* that is.

15. Lines which no where incline one to the other are called *Parallel Lines.*

16. *And two Parallel right Lines* (P, p) *have the same Inclination to a third Line* (L). For Lines parallel may be taken as one broad Line.

17. *If Parallel Right Lines be cut by a Right Line;*



1. *All the op.* ∠ *s are equal; And the contrary.* For $a = c$, $e = g$ (by 16,) but $a = b$, $c = d$, $e = f$, $g = h$, (by 10,) Therefore $a = b = c = d$; and $e = f = g = h$ by *Substitution.*

2. *The two op.* ∠ *s, whether internal or external are* = 2 ∟ s; *and the contrary.* For $f + a = b + e = c + h = d + g = 2$ ∟ *s* (by 9,); but $a = c$, $e = g$ (by 16). Therefore $f + c = b + g = a + b = e + d = 2$ ∟ *s* by *Substitution.*

18. *And, all Chords drawn parallel to the Tangent, are bisected by the Diameter passing thro' the Point of Contact.* Because they are perpendicular to it (by 12,) and their Extremities are equally distant from the Centre, and also from the Point of Contact; therefore the Parts of the Chords will be Sines of equal Arcs, and conseq. equal (by 13).

Therefore if a Chord be bisected by another at Right Angles, the bisecting one is a Diameter: And the contrary.

19. *Whence*

19. *Whence a Circumference may be drawn thro' any three Points, not in a Right Line: And the Intersection of two Right Lines bisecting the Distance between those Points at ∟s is the Centre*; Since Right Lines bisecting Chords at ∟s are Diameters (by 18.)

20. A bounded Space is called a *Figure*: If bounded by Right, or Curv'd Lines; 'tis called a *Rectilinear*, or *Curvilinear Plane Figure*.

21. Those Rectilinear Plane Figures, whose ∠s are respectively equal, and the Sides about those ∠s directly proportional, are called *Similar Rectilinear Figures*.

22. Those Rectilinear Plane Figures that are bounded by 3, or 4 Right Lines, are called *Trilateral* or *Triangular*, *Quadrilateral* or *Quadrangular Plane Figures*.

23. *Triangular Figures* are considered with respect to their *Sides* or *Angles*.

1. Those that have 3 equal, 2 equal, or 3 unequal Sides are called *Equilateral*, *Isosceles*, or *Scalenous Triangles*.

2. Those that have 1 Right ∠, 1 Obtuse, or 3 Acute ∠s are called *Right*, *Obtuse*, or *Acute* ∠d △s.

24. *Quadrilateral Figures*, whose opposite Sides are parallel, are called *Parallelograms*. And,

Right ∠d. { all } Sides equal { *Squares*.
Pgrs. having { only op. } are call'd { *Rectangles*.

An *Obliq.* ∠d. { all } Sides equal { *Rhombus*.
Pgr. having { only op. } is call'd a { *Rhomboides*.

25. And a *Parallelogram* may be conceived to be generated by a Right moveable Line drawn uniformly into the Length of a Right Immoveable one: *Therefore the Surface generated by those Lines shall contain so many little Planes equiangular with the whole, as there are Units in the Product of their Parts*.

26. *Therefore Pgrs. and consequently △s having one ∠ equal, are as the Product of the Sides about that ∠. And if those Sides be reciprocally* ∷ l, *the Parallelograms, or △s shall be equal; and the contrary*.

27. *Also*

27. *Also, if* h *and* b *be the Height and Breadth of a Rectangle; its Area will be* h b, *from its Genesis,* i. e. *'twill contain so many square Areas as there are Units in the Product of its Height and Breadth.*

And since the *Contents* of all *Surfaces* are estimated by the Square of some known *Length*; therefore such *Figures* must be reduced to *Rectangular* ones, before they can be *Measured.*

28. A Quadrilateral Figure of unequal Sides is called a *Trapezium.* And all other Rectilinear Figures, are in general, called *Polygons,* and have particular Names from the Number of their *Sides* or *Angles,* as those of 5 Sides *Pentagons,* of 6, *Hexagons,* of 7, *Heptagons,* &c.

All Plane Figures having equal, or unequal Sides or Angles, are called *Regular,* or *Irregular Figures.*

29. A Right Line drawn from ∠ to ∠ in any Figure, is called a *Diagonal,* or by some a *Diameter.*

The *Height* of any Figure is the nearest Distance between its Top and Base: *Therefore all Figures between the same Parallels have the same Height.*

30. A Plane Figure whose Extremities are equally distant from a Point therein, is called a *Circle:* And may be conceived to the generated by the Revolution of a right Line, one end being fix'd as the *Centre,* the other describing the *Circumference of the Circle.* And the Space, less than a *Semi-Circle,* described by a partial Conversion is called the *Sector of a Circle.* A Part of a Circle cut off by a Right Line is called the *Segment of a Circle.*

31. *Hence, all Circumferences, as also like Arcs, their Sines, Tangents,* &c. *are as their Radii.*

32. The *Genesis of Solids* may be exhibited various ways, some respecting the *Dimension* both of their *Solidity* and *Surface*; others, that of the *Surface* only. As,

1. A *Parallelogram* being conceived to move uniformly the Length of an immoveable Right Line, shall generate a *Solid* called a *Parallelepiped:* Or if the Describing Surface be a *Square* moving the Length of, and perpendicular to, its *Side,* the *Solid* is called a *Cube:* *Therefore a Rectangular Parallelepiped contains so many equal Cubes, as there are Units in the Product of the Square Areas* in

in the defcribing Plane, by the parts, of like meafure for Length, in the Length moved: So that if *b* be its *Height*, and *b* the *Area of its Bafe*, then *hb* = *Solid Content*; or if *l*, *b* be the *Length*, and *Breadth of the Bafe*, then *hbl* = *Solidity*.

If the defcribing Plane be a *Polygon*, or *Circle*, the generated *Solid* is called a *Prifm*, or *Cylinder*.

But if the defcribing *Polygonal*, or *Circular Plane*, in moving be fuppofed to decreafe uniformly, 'till it comes to a Point, the *Solid* generated is called a *Pyramid*, or *Cone*.

2. A Right Line moving uniformly, fo that its ends may defcribe the Periphery of two parallel, fimilar, and equal *Polygons*, will generate the Surface of a *Prifm*, whofe Bafes are thofe *Polygons*.

If the Bafes are *Pgrs.* △ s, or *Circles*, the Surface generated will be that of a *Parallelepiped*, *Triangular Prifm*, or *Cylinder*.

The right Line connecting the Centres of the *Polygonal* Bafes is called the *Axis*.

And a right Line fix'd by one end to a Point, and with the other defcribing a *Polygon*, or *Circle*, will generate the Surface of a *Pyramid*, or *Cone*, whofe Bafe is that *Polygon*, or *Circle*.

The right Line connecting the fix'd Point, and the Centre of the Bafe is alfo called the *Axis*.

And that Solid whofe *Axe* is perpendicular, or oblique to its Bafe is faid to be *Right*, or *Scalenous*.

The Converfion of a *Semicircle* round the *Diameter* will generate a Solid called a *Sphere*.

Solids contain'd under an equal Number of like *Surfaces*, are faid to be *Similar*.

33. *And Quantities, as alfo their Ratio's, that continually tend to an Equality, and therefore that approach nearer the one to the other, than any Difference that can poffibly be affign'd, do at laft become equal.* For, either they'll be at laft equal, or there will be fome Difference (*d*) nearer than which they cannot approach; But they do continu-

ally approach by *Supposition*; therefore there can at last be no difference, therefore they must be equal.

Hence, all *Curved Lines* may be considered as composed of an Infinite Number of Infinitely little right Lines: And any one of them Produced, only Touches the Curve, therefore is called a *Tangent* to that Point of the *Curve*.

And as all *Surfaces* may be consider'd as composed of an Infinite Number of Parallel, Right, or Curved *Lines*, or *Surfaces* of an infinitely small Breadth.

So also *Solids*, of an Infinite Number of Parallel, Plane, or Curved *Surfaces*, or *Solids* of an infinitely small Thickness, called the *Elementa of Figures*.

34. And by considering Quantities as generated by continual Motion, 'tis apparent, that in equal Spaces of Time, they will become greater, or less proportionally as the Celerity of the Motion by which they are so generated is greater or less: Hence the Celerity of the Motion is very properly called *Fluxion*, and the Quantity generated *Fluent*.

Now these *Fluxions* of Quantities are in the *First Ratio* of their *Nascent Augments*; and may be express'd by *Finite Quantities* proportional to them.

And as x (by moving uniformly) becomes $x + o$,

x^n becomes $\overline{x + o}|^n = x^n + n x^{n-1} \times o + \frac{1}{2}\overline{nn - n}$ $x^{n-2} \times oo +$, &c. But the

Augments o, & $n x^{n-1} o + \frac{1}{2}\overline{nn - n} x^{n-2} \; oo +$ &c.

are as 1 and $n x^{n-1} + \frac{1}{2}\overline{nn - n} x^{n-2} o +$ &c. i. e. as 1 & $n x^{n-1}$

Th. $F, x : F, x^n :: 1 : n x^{n-1}$

The *Flux.* of the *Fluents* $x, \dot{x}, \ddot{x}, \dddot{x}$, &c. are denoted by $\dot{x}, \ddot{x}, \dddot{x}, \ddddot{x}$, &c. which are the 1*st*. 2*d*. 3*d*. 4*th*, &c. *Flux.* of x.

Also, as $\ddot{x}, = F, \dot{x}$, and $\dot{x} = F, x$; So we may consider

ſider $x = F, x'$, and $x' = F, x''$, *&c.* As alſo denote the *Fluxion* of

$$\overline{ax - x^2}|^{\frac{1}{2}},\ \overset{\cdot\cdot}{\overline{ax - x^2}}|^{\frac{1}{2}},\ \overset{\prime}{\overline{ax - x^2}}|^{\frac{1}{2}},\ \frac{ax + xx}{a - x}$$

by $\overset{\cdot}{\overline{ax - xx}}|^{\frac{1}{2}},\ \overset{\cdot\cdot\cdot}{\overline{ax - xx}}|^{\frac{1}{2}},\ \overset{\prime\prime}{\overline{ax - xx}}|^{\frac{1}{2}},\ \overset{\cdot}{\frac{ax + xx}{a - x}}$

Hence $1 : nx^{n-1} :: \dot{x} : n\dot{x}x^{n-1} = F, x^n$, whether n be *Affirmative* or *Negative*, *Integer* or *Fraction*, as for Inſtance.

$$F, x^{-n} \left(\frac{1}{x^n}\right) \text{ is } \left(-n\dot{x}x^{-n-1} = -n\dot{x} \times \frac{1}{x^{n+1}} =\right) - \frac{n\dot{x}}{x^{n+1}}$$

$$\text{Th. } F, \frac{1}{x} \left(x^{-1}\right) \text{ is } \left(- \dot{x}x^{-2} = \right) - \frac{\dot{x}}{x^2}$$

$$F, x^{-\frac{m}{n}} \left(\frac{1}{x^{\frac{m}{n}}}\right) \text{ is } -\frac{m}{n}\dot{x}x^{-\frac{m-1}{n}}$$

$$F, x^{\frac{1}{2}} \text{ is } \left(\tfrac{1}{2}\dot{x}\, x^{\frac{1}{2}-1} = \tfrac{1}{2}\dot{x} \times x^{-\frac{1}{2}} = \tfrac{1}{2} \times \frac{\dot{x}}{x^{\frac{1}{2}}} = \right) \frac{\dot{x}}{2x^{\frac{1}{2}}}$$

35. In any given Equation, involving Fluents, to find the Fluxion.

Rule. *Multiply every Term of the Equation ſeparately by the ſeveral Indices of the Powers of the Fluents therein; and in every ſuch Product, change one of the Roots of the Powers into its Fluxion, the Aggregate of all the Products, connected under their proper Signs, will be the Fluxion of the Equation ſought.*

This Rule is uſually demonſtrated after this manner;

If $x^3 - xy^2 + a^2 z - b^3 = 0$; Then its *Fluxion* is $3x^2\dot{x} - \dot{x}y^2 - 2xy\dot{y} + a^2\dot{z} = 0$: For ſupp. o an

Infinitely small Quantity; let $o\dot{x}$, $o\dot{y}$, $o\dot{z}$, represent the Instantaneous Increments of the Fluents x, y, z; These after a Momentary Increment of Time, will become $x + o\dot{x}$, $y + o\dot{y}$, $z + o\dot{z}$, which being substituted instead of x, y, z, in the given Equation, 'twill be $x^3 + 3x^2o\dot{x} + 3xo^2\dot{x}^2 + o^3\dot{x}^3 - xy^2 - o\dot{x}y^2 - 2xoy\dot{y} - 2\dot{x}o^2y\dot{y} - xo^2\dot{y}^2 - \dot{x}o^3\dot{y}^2 + a^2z + a^2o\dot{z} - b^3 = 0$. From this Subduct the given Equation; Then divide the Remainder by o; and reject all Terms multiplied by it, (as being infinitely little;) There rests $3x^2\dot{x} - \dot{x}y^2 - 2xy\dot{y} + a^2\dot{z} = 0$

Also, if $x^3 - xy^2 + a^2 \times \overline{ax - y^2}|^{\frac{1}{2}} - b^3 = 0$, then its *Flux.* is $3x^2\dot{x} - \dot{x}y^2 - 2xy\dot{y} + a^2 \times \overline{a\dot{x} - y\dot{y}}|^{\frac{1}{2}} = 0$.

Or let $z = \overline{ax - yy}|^{\frac{1}{2}}$, th. $z^2 = ax - y^2$; and (by this) $2z\dot{z} = a\dot{x} - 2y\dot{y}$, th. $\dot{z} = \frac{a\dot{x} - 2y\dot{y}}{2z}$, i. e. $\overline{ax - y^2}\dot{|}^{\frac{1}{2}} = a\dot{x} - 2y\dot{y} \times \frac{1}{2\sqrt{ax - yy}}$

Th. $3x^2\dot{x} - \dot{x}y^2 - 2xy\dot{y} + \frac{a^3\dot{x} - 2a^2y\dot{y}}{2\sqrt{ax - y^2}} = 0$

And by repeating the Operation, the *Second, Third,* &c. *Fluxions* of Equations are found.

If

If $z y^3 - z^4 + a^4 = 0$, then by *Operation*

1. $\dot{z} y^3 + 3 z \dot{y} y^2 - 4 z \dot{z}^3 = 0$

2. $\ddot{z} y^3 + 6 \dot{z} \dot{y} y^2 + 3 z \ddot{y} y^2 + 6 z \dot{y}^2 y - 4 z \ddot{z}^3 - 12 \dot{z}^2 z^2 = 0$

Hence, *The Fluxion of Quantities multiplied, is the Sum of the Products of the Fluxion of each Factor, by the Product of the other Factors.* Thus,

F, xy is $\dot{x} y + x \dot{y}$; And F, by is $(b\dot{y} + 0y)\ b\dot{y}$

F, $\overline{2rx - x^2}|^{\frac{1}{2}}$ is $\left(\frac{2r\dot{x} - 2x\dot{x}}{2 \times \overline{2rx - xx}|^{\frac{1}{2}}} =\right) \frac{r\dot{x} - x\dot{x}}{\overline{2rx - xx}|^{\frac{1}{2}}}$

F, $\overline{zx + xx}|^{\frac{1}{2}}$ is $\frac{\dot{z} x + z\dot{x} + 2x\dot{x}}{2 \times \overline{zx + xx}|^{\frac{1}{2}}}$; And

F, $\overline{ax - zz}|^{\frac{3}{1}}$ is $3 \times \overline{ax - zz}|^{\frac{2}{1}} \times a\dot{x} - 2z\dot{z}$

and so in others: Also in *Fractions*

F, $\frac{x}{y}$ $\left(x \times \frac{1}{y}\right)$ is $\left(\dot{x} \times \frac{1}{y} - \frac{\dot{y}}{y^2} \times x\right) \frac{\dot{x}y - x\dot{y}}{yy}$

F, $\frac{a}{x}$ $\left(a \times \frac{1}{x}\right)$ is $\left(0 \times \frac{1}{x} - \frac{\dot{x}}{x^2} \times a =\right) - \frac{a\dot{x}}{xx}$

Th. the Flux. of any Fraction, (N *being Numerat. and* D *Denominat.*) *is* $\overline{\dot{N}D - N\dot{D}} \div DD$.

If the Indices are Flowing Quantities; as for instance,

F, y^x is $y^{x+\dot{x}} + x\dot{y}y^{x+\dot{x}-1} - y^x$, For

$y^x + \text{F}, y^x \left(= \overline{y + \dot{y}}|^{x+\dot{x}}\right) = y^{x+\dot{x}} + x\dot{y}y^{x+\dot{x}-1}$

Therefore if the Index be the *Sum*, *Product*, or *Power*, of Fluents; 'tis but substituting it, and its Flux. instead of x and $\dot{x}$ in this.

36. *And*

36. *And the Fluent* (φ) *of* $x^{\pm n}\dot{x}$ is $\frac{x^{\pm n+1}}{\pm n+1}$

The Fluent of $az^{r}\dot{z}\times\overline{b+cz^{n}}|^{m}$ is easily found to be

$$\frac{1}{mn+r+1}\times\frac{a}{c}\,(A)\times z^{r-n+1}$$

$$+A\times\frac{n-r-1}{mn+r-n+1}\times\frac{ab}{c}\,(B)\times z^{r-2n+1}$$

$$+AB\times\frac{2n-r-1}{mn+r-2n+1}\times\frac{ab}{c}\,(C)\times z^{r-3n+1}$$

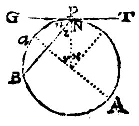
$$+\&c.\times\overline{b+cz^{n}}|^{m+1}$$

Hence φ, *of* $a\dot{y}+\overline{ay-y^2}|^{\frac{1}{2}}$ is $\frac{2}{3}\overline{ay-y^2}|^{\frac{3}{2}}$
And the like of others.

37. Where the *Extreme Value*, whether *Greatest* or *Least*, of any Quantity is required; since it is an Invariable by Sup. and its Flux. = 0; Therefore to determine it to an *Extremum*, *Put the Equation into Fluxions, let the Fluxion of that Quantity* = 0; *then will all the Terms wherein 'tis found vanish, and the Extremum be determined by the remaining ones.*

38. *The* ∠ (n,N) *made by the Tangent* (PG,PT) *and Chord* (PB) *has for its measure an Arc* = *half that subtended by the Chord.*

Draw a Diameter ⊥ PB, then is the Arc a, and Chord PB bisected (by 18), draw the Rad. CP, as also CD ∥ PB.

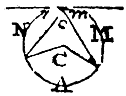


Then ∠n + ∠z = ∟ = ∠r + ∠x (by *Constr.*) But $z = x$ (by 17) th. $n = r$ by *Eq. Subduct.* And $\frac{1}{2}a$ Measures the ∠r (by 6) therefore also the ∠n. Also n + N are measured by $\frac{1}{2}a+\frac{1}{2}$A (by 9) th. N is measured by $\frac{1}{2}$A.

39. *An* ∠ (c) *in the Circumference made by two Chords, is measured by* $\frac{1}{2}$ *its subtending Arc* A.

For

For $\frac{1}{2}\overline{N + A + M}$ meafures $n + c + m$ (by 9) But $\frac{1}{2}$ N, $\frac{1}{2}$ M meafures n, m (by 38) Therefore $\frac{1}{2}$ A meafures c.



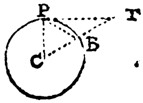
40. *All* $\angle$s *in the Circumference, fubtended by the fame Arc or Chord, are equal.* For each is meafured by $\frac{1}{2}$ the fubtending Arc.

41. *An* $\angle$ (C) *at the Centre is double that* (c) *at the Circumference ftanding on the fame Arc* (A.) For C is meafured by the whole (by 6), and c by $\frac{1}{2}$ the Arc A (by 39)

42. *An* $\angle$ *in a Segment* $>, =, <$ *Semicircle is Acute, Right, Obtufe*: For 'tis fubtended by an Arc $<, =, >$ Semicircumference, whofe $\frac{1}{2}$ (the Meafure of the $\angle$, by 39) is $<, =, > 90^{d}$. and therefore is *Acute*, *Right*, *Obtufe* (by 7)

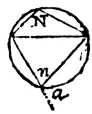
43. Whence, *To draw a Tangent* (PT) *to any point* (P) *of the Circumference of a Circle*: Let PB = *Rad.* (PC), draw CB out, let BT = BC, draw TP the *Tangent* fought.

For fince PB = BC = BT (by *Conftr.*) an Arc paffing thro' C, P, T, (by 19) will be a $\frac{1}{2}$ *Circumference*, th. $\angle$P = ∟, (by 42) th. PT is a *Tangent* to the point P (by 12)



Hence the Practical Methods of *Erecting* and *Letting* fall *Perpendiculars* are evident.

44. *Oppofite* $\angle$s (n + N) *in the Circumference, ftanding on the fame Chord are* = 2 ∟s. For each is meafured by $\frac{1}{2}$ its fubtending Arc, therefore both by $\frac{1}{2}$ the Circumference, which meafures 2 ∟s (by 7)

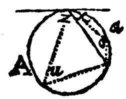


45. Hence, *The op.* $\angle$s *of a Quadrilateral Fig. infcr. in a Circle are* = 2 ∟s. *And that Quadrilat. Fig. the Sum of whofe op.* $\angle$s *is* = 2 ∟s, *may be infcrib'd in a Circle.*

46. *And the Sides thereof be produc'd, the external* $\angle$ (a) *is* = *to the internal op. one* (N.) For a (+ n = 2 ∟ (by 9) = N (+ n by this.)

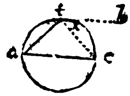
47. *The* $\angle$ (x, z,) *made by a Tangent and Chord, is* = *to an* $\angle$ (u, o) *in the op. Segment.*

For x, z, = $\frac{1}{2}a$, $\frac{1}{2}$ A (by 38) = u, o (by 39)



48. Hence

48. Hence, *To cut off a Segment from a given Circle capable of containing a given* ∠ (z).



Draw the Tangent tb (by 43,) make ∠ x = ∠ z, then $a = x$ (by 47) = z, therefore tac is the Segment reqd.

49. *Also, having the Chord* (ab) *of a Segment capable of containing a given* ∠ (x) : *To find the Point* (p) *thro' which the Arc shall pass.*

Draw ae, and at any point (e) make ∠ e = ∠ x, draw bp ∥ de, then z (= e) = x, therefore the Arc shall pass thro' p: In the same manner the other points are found.

50. *In a Circle, equal Chords are equally distant from the Centre; and the Contrary.* For those = *Chords* are = double *Sines*, therefore their *Co-sines*, which is their distance from the Centre, must be =.

51. *In a Triangle, the* ½ *of each Side is the Sine of its op.* ∠. For (being inscr. in a Circle by 19) the Sides are Chords of Arcs, which measure ½ their op. ∠ s; And ½ those Chords are the Sines of those Arcs (by 8) *i. e.* ½ the Sides are the Sines of their op. ∠ s.

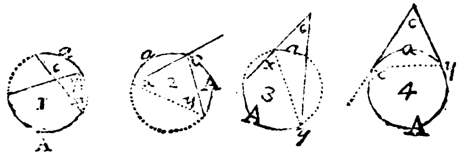
52. *The* 3 ∠ s *of every Plane* △, *is* = 2 ∟s: For (being inscr. in a Circle by 19) their Measure is ½ the Circumference, which measure 2 ∟*s* (by 7)

53. *Therefore in a Plane* △, *if the Side be produced, the extern.* ∠ (d) *will be = to the two intern. and op. ones* ($a + b$.) For $a + b$ ($+ c$ = 2 ∟s by 52) = d ($+ c$ by 9)

54. *Hence the measure of an Angle*, as c in *Figure*

1. *Neither at Centre or Circumf. made by two Crossing Chords*; Or.

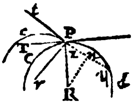
2. *In the Circumf. made by the Chord and Secant*; *is* (= $y + x$ by 53) = ½ A + ½ a (by 39)



3. *With-*

3. *Without the Circumference, made by two Secants*; or 4. *Made by two Tangents*; *is* ($= x - y$ by 53) $=$ $\frac{1}{2}A - \frac{1}{2}a$ (by 39)

55. *The mixt* ∠ RPd *made by the Circumference, and Rad. is greater than any Rectilinear Acute* ∠ (RPy *or* z.) For drawing Rn Perp. Py, then ∠n (= ∟) > ∠z. therefore RP > Rn, and the point n is within the Circle Conseq. ∠z < ∠RPd.



Whence, the ∠ made by the Circumference and Tangent is less than any Acute ∠.

Therefore no Right Line can be drawn between the Tangent and Circumference, tho' an infinite Number of Curv'd ones may.

56. *A Curvilinear ∠* (CPc) *made by two Intersecting Circumferences* (C, c) *is = to a Rectilinear ∠* (RPr) *made by the Radii* (RP, rP) *drawn to the Intersect. point* (P.) For, drawing the Tangents PT, Pt, then ∠cPC = ∠tPT, and tPT (+ TPr = ∟) = RPr (+ rPT.)

57. *In a Triangle* >, =, < *Angles are subtended by* >, =, < *Sides*; *And the contrary.* For >, =, < Sides are Chords of Arcs which measure >, =, < Angles.

1. *Therefore Equilateral* △*s must be also Equiangular*; *And the contrary.*

2. *And those* △*s are equal, if in both, either 3 Sides*; *or 2 Sides, with the included ∠*; *or 2 Sides, with the ∠ op. to the same Side*; *or 2∠s, with the interjacent Side*; *or 2∠s, with the Side subtending the same ∠*; *be equal.*

3. *Therefore the ∠s at the Base of an Isosceles* △ *are* =; For the equal Sides are Chords of Arcs measuring = ∠s.

4. *And that Line from the Vertex bisecting the Base of an Isosceles* △, *is Perp. to it.*

58. *The Intern. ∠s of any Polygon are = to twice as many* ∟*s as it has Sides, except 4.* For (n being the N° of Sides or ∠s,) every Polygon may be divided into $n - 2$ △s, (by Lines drawn from any ∠ to all the rest, except the adjoining ones,) therefore will have $2n - 4$ ∟s.

H h 1. *And*

1. *And (producing the Sides) all the extern* ∠*s* = 4 ∟*s*. For each Intern with its Extern is = 2 ∟s, (by 9) therefore all the Intern and Extern ∠s = 2 *n* ∟s; but the Intern ∠s = 2 *n* ∟ — 4 ∟s (by 58) th. the Extern ∠s = 4 ∟s.

2. *Therefore the Intern* ∠ (A) *of any Regular Polygon is* $= \overline{\overline{2n - 4}} \div n$ ∟; For nA $= \overline{2n - 4}$ ∟s

3. *And the* ∠ (a) *at the Centre is* = 4 ∟s ÷ *n* (= 2 ∟s — A)

59. *Hence it follows, that there are only 3 Regular Surfaces that can fill a Space*, viz. *Triangles, Squares, and Hexagons.*

Since all the ∠s round a Point = 4 ∟s, and the ∠ of a {*Regul.* △ is $\frac{2}{3}$ ∟, *Square* is 1 ∟, *Hexagon* is $\frac{4}{3}$ ∟} which multiply by {6, 4, 3} = 4 ∟.
But in a *Pentagon*, 3∠s are *less*, 4∠s *greater* 4 ∟s, and in Figures having more Sides, 3 ∠s *greater* 4 ∟s, therefore, &c.

60. *Also, There are only 5 Regular Bodies.* For a Solid ∠ consists of less than 4 ∟s, and 3 Plane ∠s are the fewest that can make it; But 6, 4, 3 ∠s of △s, □s, *Hexagons*, make 4 ∟s, and 4∠s of *Pentagons* are greater 4 ∟s, therefore only a △, □, and a *Pentag.* can form a Solid ∠; so that there can be but 5 *Regular Bodies*, viz. the

	Solid ∠*s*	made of	And has *Plane Sides*	*Lin. Edges*	*Plane* ∠*s*
Tetr-aedrum of	4	3 △s	4	6	12
Oct-aedrum of	6	4 △s	8	12	24
Icos-aedrum of	12	5 △s	20	30	60
Hex-aedrum of	8	3 □s	6	12	24
Dodec-aedrum of	20	3 ⬠s	12	30	60

61. *The Intersection of 2 Planes is a Right Line*: For the Right Line drawn on either Plane from any two Points of the Section will be the same. And

1. *The Intersections of 2 Paral. Planes by a 3d are Paral.*
2. *A Perp. to crossing Lines, is Perp. to their Planes.*

3. *Right*

3. *Right Lines, or Planes, Perp. to the same Plane, or Right Lines are Parallel.*

4. *A Perp. to a Plane is Perp. to its Paral. Plane; and the contr.*

5. *If a Right Line* (L) *be Perp. to any Plane* (P) *then all Planes passing thro'* (L) *are Perp. to* P; *And if 2 Planes be Perp. to* P, *their Intersection will also be Perp. to* P.

6. *In any Number of Planes passing thro' the Line* L *Paral. to a certain Plane* P, *their Intersection with* P *are* || *to* L, *and to each other.*

7. *Two pair of meeting Lines respectively* || *in different Planes contain* = ∠s, *and Paral. Planes.*

62. *In Equiangular* △ s, *the Sides about* = ∠ s *are directly* : : l; *And the contr.* For, being inscr. in a Circle, the Sides will be Chords of Similar Arcs.

63. *Therefore Equiangular* △ s *are similar* (by 21); *Conseq. similar* △ s *have their Sides about* = ∠ s : : l.

1. *And therefore those* △ s *are also similar, which have an* ∠ =, *and the Sides about that* ∠ : : l; *Or which have 2 Sides* : : l, *an* ∠ *op. to one of 'em* =, *and the other of the same kind.*

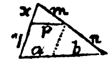
2. *Hence similar* △ s *have their Heights* : : l *to their Bases*: For making a Side next the vertical ∠ Radius, the Heights will be Sines of similar Arcs, which are as the Radii (by 31.) and those Radii as the Bases (by 62).

3. *And a Right Line, or Plane cutting a Triangle, or Pyramid, either parallel or subcontrary to the Base, cuts off a Figure similar to the whole.* For the ∠ at the Top is common, the others are = by *Constr.* therefore the Figures are similar.

4. *Conseq. in a* △, *the parts of the Sides cut by a Right Line drawn* || *to its Base are directly* : : l; *And the contr.* For the △ cut off being Sim. to the whole (by *Constr.*) the Sides about the common ∠ at the Top will be : : l (by 62.) therefore, &c.

5. Hence is drawn the Method of dividing a Right Line in a given : : ; And of finding a 3d, or 4th : : l to 2, or 3 given right Lines; with Variety of Practices in *Geometry.*

6. *In any* $\triangle$, *the Base is to its Parallel, as the Sides are to the Parts next the Vertex.*



For $y : x :: n : m :: b : p$ (a) by 4. 63
Th. $y + x : x :: n + m : m :: b + a : p$

7. *In any Quadrilat. Fig. inscrib'd in a Circle, the* $\square$ *of the Diagonals is* = *to the* 2 $\square$ s *of the op. Sides.*

Let $\angle$ DAE = $\angle$ BAC, AC = y, BE = m; then $\triangle$s AED, ABC, and ABE, ACD, are *Similar.*

Th. $\left\{\begin{matrix} b : n :: y : c \\ a : m :: y : d \end{matrix}\right\}$ by 63, th. $\left\{\begin{matrix} bc = ny \\ ad = my \end{matrix}\right\}$

Th. $ad + bc = (my + ny)\ y \times \overline{m+n}$

8. *All Figures of the same kind, standing upon the same, or equal Bases, and of* = *Altitudes are* =. For the *Elementa* of the one are respectively equal to those of the other, both in Number and Magnitude. Th. in *Pgrs. Ppds. Prisms*, and *Cylinders*, if B, H, be the *Base*, and *Height*, then BH = *Content*: And in *Triangles*, the *Area* = $\frac{1}{2}$ BH; bec. *Triangle* : *Circumscr. Pgr.* :: 1 : 2 (by this).

9. And a Rectilinear Figure of n Sides may be reduced into $n-2$ *Triangles*, by $n-3$ *Diagonals*; and its *Area* is = to the Area of all its Triangles.

10. Th. in a Circle whose *Area* call A, Circumf. c, Rad. r, since $\frac{1}{2} r c$ ($\frac{1}{2} rc^{\circ} c$) = A, th. $\frac{1}{2} rc$ = A, (by 36.) And d^2 ($4r$) : $\frac{1}{2} rc$ :: $2r$ (d) : $\frac{1}{2} c$. Also, if c = *Arc*, then the *Sector* = $\frac{1}{2} rc$.

11. *All Homologous Figures,* (1) *Of equal Bases are as their Heights, or of equal Heights, are as their Bases;* (2) *Of different Bases and Heights, are as their Products, or those of the Sides into the Bases or Heights;* (3) *And theref. are also as their like Sides raised into the Figure's Dimension.*

Suppose L, l, the *Length* or *Sides* of the Figures (x, y) or let S, rS, ϱS, the Sides of the one, and s, rs, ϱs, those of the other; then, if

If $\left\{\begin{matrix} B = b, \text{ or } H = h;\ x : y :: BH : bh :: H : h, \text{ or } :: B : b. \\ \text{all differ};\ x : y :: BH : bh :: BL : bl :: HL : hl. \end{matrix}\right.$

For B : b :: H : h (by 2. 63) :: L : l. (by 63) Therefore

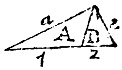
In

In $\begin{cases} \textit{Planes } x:y::(S^2r:S^2r::)\ S^2:S^2. \\ \textit{Solids } x:y::(S^3r\varrho:S^3r\varrho::)\ S^3:S^3. \end{cases}$

And like infcrib'd Figures are as their refpective wholes, *Th. all fimilar Surfaces, or Solids, are as the Squares, or Cubes of their like Sides.*

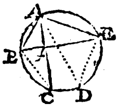
And if fuch Figures are equal, they have their Bafes and Heights reciprocally : : l. *Aud the contrary.*

12. *A Right Line bifecting an* ∠ *of a* △, *cuts the op. Side directly* : : l *to the Legs.* i.e. $y:z$ (: : A : B (by 11.63) : : $ax:ex$ (by 26) : : $a:e$.



13. *If from any* ∠ (BAE) *of a* △ *Infcr. in a Circle, whofe Diameter is* AD, *a Perp.* (AF) *be let fall upon the op. Side; then* (*bec.* △ ABF *Sim.* AED) AF : AB : : AE : AD.

Alfo, let AC bifect ∠ A, then (bec. △ ABC *Sim.* AEF) AC : AB : : AE : AF, or BAE = CAF = AF^2 + CFA = AF^2 + BFE.

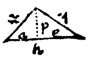


In any Plane △, whofe *Sides* are a, b, c, *Height* p, and their op. ∠*s* x, y, z. Since the *Diameter* (of the Circumfcr. Circle) $d:a::c:(ac \div d)\ p$. th. Area A $= acb \div 2d$, th. $d:\frac{1}{2}a::cb$: A, or d : s, ∠ $x::cb$: A.

Again, △*s having one* ∠ *common, are as the Facts of the Sides about that* ∠.

Alfo, fince $d:c::(ab:pb = 2\text{A}::)\ \frac{1}{4}ab:\frac{1}{2}\text{A}$, or $\frac{1}{2}d:c::\frac{1}{4}ab:\text{A}$, th. $\frac{1}{2}cd:c^2::\frac{1}{4}ab:\text{A}$, that is, $d\times$s, z : c^2 : : s, x × s, y : A.

64. *A Perpendicular demitted from the Right* ∠ *of a Plane* △ *upon the op. Side, divides that* △ *into two, each fimilar to the whole, and therefore to one another.* For each has 1 ∟, and 1 ∠ common to the whole, therefore are fimilar (by 63.) Whence



1. $\begin{cases} a:x::x:b \\ e:y::y:b \end{cases}$ by 63, $\begin{cases} \text{th. } x^2 = ab \\ \text{th. } y^2 = eb \end{cases}$

2. Th. $x^2 + y^2\ (= ab + eb = \overline{a+e}\times b) = b^2$

3. In a ∟△, $b = \overline{x^2+y^2}|^{\frac{1}{2}}$, $x = \overline{b^2-y^2}|^{\frac{1}{2}}$, $y = \overline{b^2-x^2}|^{\frac{1}{2}}$

4. And

4. And $x^2 : y^2 (:: ab : eb) :: a : e$.

5. $x^2 : p^2 (:: ab : ae) :: b : e$, & $y^2 : p^2 (:: eb : ae) : b : a$

6. $b + y : x :: x : b - y$; And $b + x : y :: y : b - x$.

7. $a : p :: p : e :: x : y$, th. $p^2 = ae$; $ay = px$; $ex = py$.

8. Th., *To find a mean* :: l *between any 2 Right Lines* a, e.

Let $a + e$ be made the Diameter of a Circle, the Perp. erected on the Point of meeting, and terminated by the Circumf. will be the mean reqd.

9. *Whence we have the Method of making a Square* = *to a given Pgr. And th.* = *to any Right lin'd Figure given.*

10. *Also, To cut a Line* (l) *in Extream and Mean* :: ; (*i.e. that the* □ *of the greater part* (x) *may be* = *to the* □ *made of the lesser part* (l — x) *and the whole Line.*)

Make l and $\frac{1}{2} l$ the Legs of a ∟ △; then the *Hypotenuse* $- \frac{1}{2} l = x$ sought. For $x^2 = l^2 - lx$, th. $x^2 + lx = l^2$, th. $x = \overline{l^2 + \frac{1}{4} l^2}|^{\frac{1}{2}} - \frac{1}{2} l$.

11. And, In any Plane △, the Sides being b, x, y : Sup. lines (m, m) from the Vertex v, making each at the Base b a like ∠ with that at v; let n be the part of the Base line Intercepted between m and m, Then $b^2 = x^2 + y^2 \pm \frac{n}{m} xy$, if $v >$, or $<$ 90 *degrees.*

Theref. if $v = 90°$, $b^2 = x^2 + y^2$; If $v = 120°$ or $60°$, $b^2 = x^2 + y^2 \pm xy$:

Hence the Radius of any Circle is = *Chord of* 60°.

If $v = 135°$, or $45°$, $b^2 = x^2 + y^2 \pm xy \times \sqrt{2}$

If $v = 150°$, or $30°$, $b^2 = x^2 + y^2 \pm xy \times \sqrt{3}$, &c.

12. *In a Circle, the Circumscrib'd Square is to the Inscrib'd as* 2 *to* 1. For the (*Diameter*, or) *Side* of the Circumscr. is the *Diagonal* of the Inscrib'd Square; but the Diagonal squared is to the Side squared, as 2 to 1.

Hence

13. *Hence the Diagonal* (D) *is Incommenfurable to the Side* (S) *of the Square.* For S : D :: 1 : $\sqrt{2}$, and (s) any part of S fhall be in the fame :: to d, its correfponding part of D; and how fmall foever s be taken, yet d is capable of a further Divifion, and that *in infinitum*; therefore 'tis impoffible to find fuch a part s, that will exactly meafure D.

Whence the Infinite Divifibility of Quantity is evident.

14. *If fimilar Figures* (H, X, Y) *be made on the Sides* (h, x, y) *of a* $\perp \triangle$; *that* (H) *on the Hypotenufe* (h) *is = thofe* (X, Y) *on the Legs* (x, y.) For H, X, Y are as h^2, x^2, y^2, (by 11.63) But $h^2 = x^2 + y^2$, (by 2.64) th. H = X + Y.

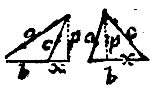
Hence 'tis eafie to find the Difference between any fimilar Figures; As alfo, the Sum of any Number of fuch Figures.

And the Quadrature of the Lunes *of* Hyppocrates *of* Scio, *is alfo evident.* Let abc be an *Ifofceles* $\triangle$, $\llcorner$ at a. Then $m + r = x + r$ (by 4.64) th. $m = x$.

15. *Alfo fimilar Figures made on* 4 :: l *Lines* (a, b, c, d) *are* :: l. As fup. *Sim.* $\triangle s$ A, B on a, b; and *Sim. Pentag.* C, D on c, d; then A : B :: $a^2 : b^2$ (by 11.63) :: $c^2 : d^2$ (by *Proport.*) :: C : D (by 11.63)

16. *In Obtufe, or Acute* $\angle$d $\triangle$s. $c^2 = a^2 + b^2 \pm 2bx$.

For ($p^2 =$) $a^2 - \overline{b^2 \pm 2bx + x^2}$ $= c^2 - x^2$



17. *Hence* $x = \pm\, \overline{a^2 \mp c^2 \mp b^2} \div 2b$

18. *Alfo, given the* 3 *Sides of a Plane* $\triangle$, *reqd. the Area.* (Fig. 2.) Since $p^2 = c^2 - x^2 = a^2 - b^2 + 2bx - x^2$, th.

$$x = \overline{c^2 + b^2 - a^2} \div 2b, \ \& \ p = \overline{c^2 - x^2}\big|^{\frac{1}{2}}. \text{ th.}$$

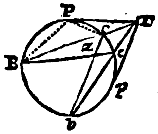
$$(\tfrac{1}{2} b \times p \text{ or}) \ \overline{\tfrac{1}{4} b^2}\big|^{\frac{1}{2}} \times \overline{c^2 - \overline{\overline{c^2 + b^2 - a^2} \div 2b}\big|^{\frac{2}{1}}}\Big|^{\frac{1}{2}} \text{ or}$$

$$\overline{\tfrac{1}{8}\, \overline{a^2 b^2 + a^2 c^2 + b^2 c^2} - \tfrac{1}{16}\, \overline{a^4 - b^4 - c^4}}\Big|^{\frac{1}{2}} \text{ or}$$

$$(\text{If } h = \tfrac{1}{2}\overline{a + b + c}) \ \overline{h \times \overline{h - a} \times \overline{h - b} \times \overline{h - c}}\big|^{\frac{1}{2}} = \textit{Area.}$$

19. *A*

19. *A Tangent* (PT) *is a mean* : : l *between the whole Secant* (BT) *and its extern. Part* (CT.)
For △ TPC *Sim.* △ BTP, theref. BT : PT : : PT : CT, or $\overline{PT}^2$ = BTC.



20. *Whence the* ▯s *made of such Secants, from the same Point* (T) *and their outward Segments are* =. i. e. BTC = *b*T*c*, For each is = to the same □ (PT^2.)

21. *Theref. those Secants are reciprocally as their outward Segments*; i. e. BT : *b*T : : *c*T : CT.

22. Hence also, *Two Tangents drawn from the same point are* = : i. e. PT = *p*T; For the Sq. of each is = to the same Rectangle.

23. *And the parts of Crossing Chords are reciprocally* : : l: i. e. Bz : Cz : : *b*z : *c*z, or Bz*c* = *b*zC; For △ BzC *Sim.* △ *b*z*c*.

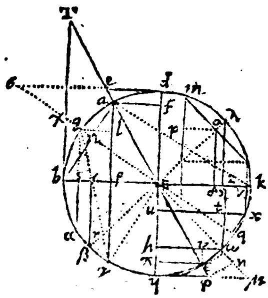
65. Hence if *r* = *Rad.* d = *Diameter*, s = *Sine*, v = *Versed-sine*, *v* = *Co-versed-sine*, c = *Chord*, *s* = *Co-sine*, t = *Tangent*, *t* = *Co-tangent*, ſ = *Secant*, *ſ* = *Co-secant*, *a* = *Arc*.

1. $dv - vv = s^2 = 2rv - v^2$; for $d - v : s :: s : v$.

2. $d : c :: c : v$; for $dv\ (= v^2 + s^2) = c^2$.

3. $C^2 : c^2\ (:: dV : dv) :: V : v$.

4. $d : C + c :: C - c : V - v$, for $C^2 - c^2 = dV - dv$



5. $tt = r^2$; for BT×DE = BCD

6. $sſ = r^2$; for Aϱ×CE = ACD

7. $sſ = r^2$; for Cϱ×CT = BCA

8. Th. $t,A \times t,A\ (= r^2) = t,a \times t,a$

9. And $s,A \times ſ,A\ (= r^2) = s,a \times ſ,a$

10. $sr = st$; for Aϱ × CB = Cϱ ×BT

11 $\begin{cases} rv = s,\tfrac{1}{2}a \times 2s,\tfrac{1}{2}a; & \text{for } CBϱ = BN \times BA \\ rv = s,a \times t,\tfrac{1}{2}a & \text{for } CBϱ = Aϱ \times B\chi \end{cases}$

12. Th. $\tfrac{1}{2}r\,v = s^2,\tfrac{1}{2}a = \tfrac{1}{2}s,a \times t,\tfrac{1}{2}a$

13. And

13. And L,v = 2 L,s,$\frac{1}{2}$ *a* — L,s, 30°

14. $\left\{\begin{array}{l}\frac{1}{2}rv = s^2, \frac{1}{2}a;\ \text{for } \frac{1}{2}K\varrho \times CB = CN.(\frac{1}{2}AK) \times CN \\ \frac{1}{2}rv = \frac{1}{2}s, a \times t, \frac{1}{2}a;\ \text{for } \frac{1}{2}K\varrho \times CD = \frac{1}{2}A\varrho \times D\sigma\end{array}\right.$

15. $\left\{\begin{array}{l}r \times s, 2a = 2s, a \times s, a;\ \text{for CB} \times A\varrho = AB \times CN \\ r \times v, 2a = 2s^2, a;\ \text{that is, CB} \times B\varrho = AB \times BN\end{array}\right.$

16. Alſo,ſ,*a* + or —t,*a* = *t* or $\overline{\text{t},\frac{1}{2}90° - a}$

17. And 'tis evident, Cε × (Vε) Hπ = πε × εω, for ΔεVω Sim. ΔεπC; But taking the Tangent εω infinitely ſmall, 'twill be = its correſponding Arc, th. s × εω = r × Hπ; *Conſeq. the Sum of all the Sines erected on any Arc* (εY) *is* = rv: *Th. the Sum of all the Sines erected on the Quadrantal Arc is* = r².

18. Alſo, if εω be infinitely ſmall, then Cφ : CY : : φμ ($\dot{t}$) φn, and Cφ : Cε : : φn : εω ($\dot{a}$), Th.$\dot{ſ}$² : r² : : $\dot{t}$: $\dot{a}$

19. Cλ : CO : : λs : P*ι* (Oδ); & Cλ : Cs : : MO : MP

Th. (P*ι* ± MP) s,$\overline{A \pm a}$ = $\overline{s,a \times s, A \pm s, A \times s, a}$ ÷ r

20. *If*, kq = 30°, *and* qx = qω, *then* yq = 60°. And ωz = xv + $\sqrt{4\,qx^2 \text{ (or } 4 \times \frac{1}{2}\omega x^2 = 4\,xt^2) - xt^2\ (qx^2)}$ or √3 qx², or √3 × qx, *i. e.* s, *Arc* > 30° = s, *Arc as much* < 30° + s, *of its Defect* × √3.

21. *Alſo* ux = hω + ωq, *i. e*, s, *Arc* > 60° = s, *Arc as much* < 60° + s, *of its Defect*.

22. If Bα, Bβ, Bγ, be *Equidiff. Arcs*, which call *a*, *m*, *e*; their common Diff. *d*, then Cγ : CR : : γG (2τβ) : : γL : ϱγ + sα, *i. e. r* : *s*,*d* : : 2s,*m* : s,*e* + s,*a* = ʐ

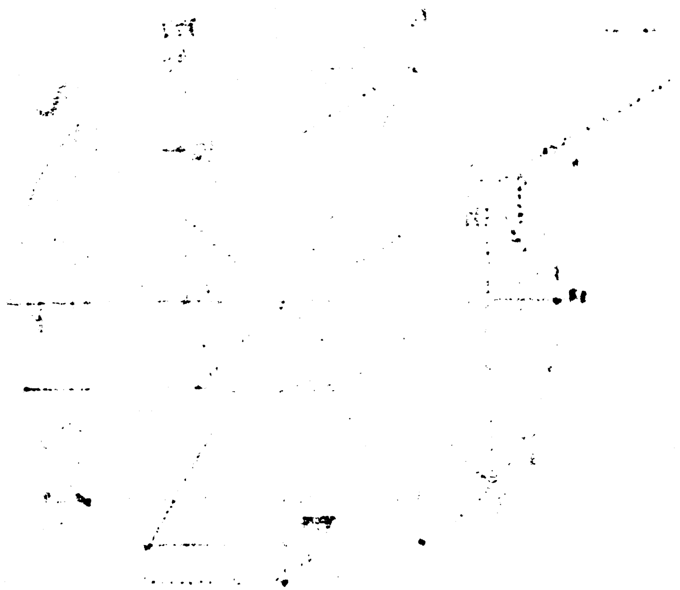
Th. s,*m* : s,*a* + s,*e* : : s,*M* : s,*A* + s,*E*

And s,*a* + s,*e* = $\overline{s,d}$ × $\overline{2s,m}$ or 2 *s*,*d* × s,*m* ÷ r

23. Becauſe *w* = S—s, mo = $\frac{1}{2}$*Chord* of $\overline{A—a}$, then op, produc'd to dc, will be = $\frac{1}{2}\overline{S+s}$; Th. δ (V—v) : S—s : : $\frac{1}{2}\overline{S+s}$: $\overline{S^2 - s^2}$ ÷ $\frac{1}{2}$δ (q), and q ± $\frac{1}{2}$δ = *s*,*a*, or *s*,*A*

24. Since $\overline{r^2 - s^2}|^{\frac{1}{2}}$ (*s*) : *r* : : $\dot{s}$: ($\dot{a}$) $\overline{r^2 - s^2}|^{-\frac{1}{2}} \times r\,\dot{s}$

$$\text{Th. } a = \frac{s}{1} + \frac{1 \times s^2}{2 \times 3r^2}\alpha + \frac{9 \times s^2}{4 \times 5r^2}\beta + \frac{25 \times s^2}{6 \times 7r^2}\gamma + \&c.$$



13. And $L,v = 2\,L,s,\frac{1}{2}a - L,s,\ 30°$

14. $\begin{cases} \frac{1}{2}rv = s^2,\frac{1}{2}a; \text{ for } \frac{1}{2}K\varrho \times CB = CN\,(\frac{1}{2}AK) \times CN \\ \frac{1}{2}rv = \frac{1}{2}s,a \times t,\frac{1}{2}a; \text{ for } \frac{1}{2}K\varrho \times CD = \frac{1}{2}A\varrho \times D\sigma \end{cases}$

15. $\begin{cases} r \times s,2a = 2s,a \times s,a; \text{ for } CB \times A\varrho = AB \times CN \\ r \times v,2a = 2s^2,a; \text{ that is, } CB \times B\varrho = AB \times BN \end{cases}$

16. Also, $s,a + \text{or} - t,a = t \text{ or } t,\overline{\tfrac{1}{2}\,90° - a}$

17. And 'tis evident, $C\varepsilon \times (V\varepsilon)\ H\pi = \pi\varepsilon \times \varepsilon\omega$, for $\Delta\varepsilon V\omega$ Sim. $\Delta\varepsilon\pi C$; But taking the Tangent $\varepsilon\omega$ infinitely small, 'twill be = its corresponding Arc, th. $s \times \varepsilon\omega = r \times H\pi$; *Conseq. the Sum of all the Sines erected on any Arc* (εY) *is* $= rv$: *Th. the Sum of all the Sines erected on the Quadrantal Arc is* $= r^2$.

18. Also, if $\varepsilon\omega$ be infinitely small, then $C\varphi : CY :: \varphi\dot{\mu}\ (\dot{t})\ \varphi n$, and $C\varphi : C\varepsilon :: \varphi n : \varepsilon\omega\,(\dot{a})$, Th. $s^2 : r^2 :: \dot{t} : \dot{a}$

19. $C\lambda : CO :: \lambda s : P\iota\,(O\delta)$; & $C\lambda : Cs :: MO : MP$ Th. $(P\iota \pm MP)\,s,\overline{A \pm a} = \overline{s,a \times s,A \pm s,A \times s,a} \div r$

20. *If*, $kq = 30°$, *and* $qx = q\omega$, *then* $yq = 60°$. And $\omega z = \overline{xv + \sqrt{4\,qx^2}\ (\text{or } 4 \times \tfrac{1}{2}\omega x^2 = 4\,xt^2) - xt^2\,(qx^2)}$ or $\sqrt{3\,qx^2}$, or $\sqrt{3} \times qx$, *i.e.* s, *Arc* $> 30° =$ s, *Arc as much* $< 30° +$ s, *of its Defect* $\times \sqrt{3}$.

21. *Also* $ux = h\omega + \omega q$, *i. e,* s, *Arc* $> 60° =$ s, *Arc as much* $< 60° +$ s, *of its Defect.*

22. If $B\alpha$, $B\beta$, $B\gamma$, be *Equidiff. Arcs*, which call a, m, e; their common Diff. d, then $C\gamma : CR :: \gamma G\ (2\tau\beta) :: \gamma L : \varrho\gamma + s\alpha$, *i. e.* $r : s,d :: 2s,m : s,e + s,a = z$

Th. $s,m : s,a + s,e :: s,M : s,A + s,E$

And $s,a + s,e = s,d \times 2s,m$ or $2\,s,d \times s,m \div r$

23. Because $w = S - s$, $mo = \frac{1}{2}$*Chord* of $A - a$, then op, produc'd to dc, will be $= \frac{1}{2}\overline{S + s}$; Th. $\delta\ (V - v) : S - s :: \frac{1}{2}\overline{S + s} : \overline{S^2 - s^2} \div \frac{1}{2}\delta\ (q)$, and $q \pm \frac{1}{2}\delta = s,a$, or s,A

24. Since $\overline{r^2 - s^2}|^{\frac{1}{2}}\ (s) : r :: \dot{s} : (\dot{a})\ \overline{r^2 - s^2}|^{-\frac{1}{2}} \times r\,\dot{s}$

Th. $a = \frac{s}{1} + \frac{1 \times s^2}{2 \times 3r^2}\,\alpha + \frac{9 \times s^2}{4 \times 5r^2}\,\beta + \frac{25 \times s^2}{6 \times 7r^2}\,\gamma + \&c.$

 25. And

25. And $s = \frac{a}{1} - \frac{a^3}{1 \times 2 \times 3r^2} + \frac{a^5}{1 \times 2 \times 3 \times 4 \times 5r^4} -$, &c.

26. Bec. $\overline{dv - v^2}|^{\frac{1}{2}}$ (s) : r : : $\dot{v}$: ($\dot{a}$) $\overline{dv - v^2}|^{-\frac{1}{2}} \times r\dot{v}$

Th. $a = \overline{1 + \frac{v}{6} + \frac{3v^2}{40d} + \frac{5v^3}{112d^2} + \frac{35v^4}{1152d^3}, \text{\&c.}} \times \overline{dv}|^{\frac{1}{2}}$

27. $v = \frac{a^2}{1 \times 2r} - \frac{a^4}{1 \times 2 \times 3 \times 4r^3} + \frac{a^6}{1 \times 2 \times 3 \times 4 \times 5 \times 6r^5}$, &c.

28. Th. $s(r - v) = r - \frac{a^2}{1 \times 2r} + \frac{a^4}{1 \times 2 \times 3 \times 4r^3} -$, &c.

29. ſ$(1 \div s) = 1 + \frac{1}{2}a^2 + \frac{5}{24}a^4 + \frac{61}{720}a^6 + \frac{277}{8064}a^8 +$, &c.

30. Th. S,ſ's $= a + \frac{1}{6}a^3 + \frac{1}{24}a^5 + \frac{61}{5040}a^7 + \frac{277}{72576}a^9$, &c.
$=$ L, t, $\overline{\frac{1}{2}90 + a}$ in *Naper*'s Form.

31. $\int (1 \div s) = 1 \div a + \frac{1}{6}a + \frac{7}{360}a^3 + \frac{31}{15120}a^5 +$, &c.

32. Bec. $\frac{r^2}{s} =$ ſ, & $\frac{r\dot{s}}{s} = \dot{a}$, alſo ($\dot{a}$ſ) $\frac{r^3\dot{s}}{r^2 - s^2} =$ F,S,ſ's

$= \dot{s} + \dot{s}s^2 + \dot{s}s^4 + \dot{s}s^6 + \dot{s}s^8 + \dot{s}s^9 +$, &c. (if $r = 1$)
Th. S, ſ's $= s + \frac{1}{3}s^3 + \frac{1}{5}s^5 + \frac{1}{7}s^7 + \frac{1}{9}s^9 + \frac{1}{11}s^{10}$, &c.
Alſo S, t's $= \frac{1}{2}s^2 + \frac{1}{4}s^4 + \frac{1}{6}s^6 + \frac{1}{8}s^8 + \frac{1}{10}s^{10}$, &c.

33. And (if $r = 1$) $\dot{a} = \dot{t} \div \overline{1 + t^2}$ (by 18, 65) $=$
$\dot{t} \times \overline{1 - t^2 + t^4 - t^6 + t^8 - t^{10} + t^{12}\ \text{\&c.}}$
Th. $a = t - \frac{1}{3}t^3 + \frac{1}{5}t^5 - \frac{1}{7}t^7 + \frac{1}{9}t^9 - \frac{1}{11}t^{11} +$, &c.

34. And $t = a + \frac{1}{3}a^3 + \frac{2}{15}a^5 + \frac{17}{315}a^7 + \frac{62}{2835}a^9 +$, &c.
Th. $t\,(1 \div t) = 1 \div a - \frac{1}{3}a - \frac{1}{45}a^3 - \frac{2}{945}a^5 - \frac{1}{4725}a^7 -$, &c.

35. S,t's $= \frac{1}{2}a^2 + \frac{1}{12}a^4 + \frac{1}{45}a^6 + \frac{17}{2520}a^8 + \frac{31}{14175}a^{10}$, &c.

36. And if $n = 2,3058509$, &c. (10 being *Radius*) then
$n \pm \frac{1}{2}a^2 \pm \frac{1}{12}a^4 \pm \frac{1}{45}a^6 \pm \frac{17}{2520}a^8$, &c. $=$ L,ſ, or L,s, of a, in *Naper*'s Form; And $\frac{1}{n}$th thereof will give *Briggs*'s.

37. Given c the Chord of an Arc a: Reqd. C that of another Arc A, ſo that $A : a :: n : 1$. Since

$a = s +$

$a = c + \frac{c^3}{6d^2} + \frac{3c^5}{40d^4}$, &c. $A = C + \frac{C^3}{6d^2} + \frac{3C^5}{40d^4}$, &c. (by 24)

Th. $C + \frac{C^3}{6d^2} +$, &c. $= n \times \overline{c + \frac{c^3}{6d^2} +, \text{\&c.}} = A$

Th. $C = \frac{nc}{1} + \frac{1 - n^2}{2 \times 3d^2} c^2 \alpha + \frac{9 - n^2}{4 \times 5d^2} c^2 \beta + \frac{25 - n^2}{6 \times 7d^2} c^2 \gamma$, &c.

38, Bec. t, $a = (\frac{rs}{s})\ \frac{rs}{\sqrt{r^2 - s^2}} =$ (if a be 30°) $\frac{\frac{1}{2}}{\sqrt{\frac{3}{4}}} = \frac{1}{\sqrt{3}}$

And $6a$, or $6 \times \overline{t - \frac{1}{3}t^3 + \frac{1}{5}t^5, \text{\&c.}} = \frac{1}{2}$ Periphery (π)

But $6 \times \frac{1}{\sqrt{3}} = \frac{\sqrt{36}}{\sqrt{3}} = \sqrt{12} = 2\sqrt{3}$, and $t^2 = \frac{1}{3}$; Let $\alpha = 2\sqrt{3}$, $\beta = \frac{1}{3}\alpha$, $\gamma = \frac{1}{3}\beta$, $\delta = \frac{1}{3}\gamma$, &c.

Then $\alpha - \frac{1}{3}\beta + \frac{1}{5}\gamma - \frac{1}{7}\delta + \frac{1}{9}\varepsilon$, &c. $= \frac{1}{2}\pi$, or

$\alpha - \frac{1}{3}\frac{3\alpha}{9} + \frac{1}{5}\frac{\alpha}{9} - \frac{1}{7}\frac{3\alpha}{9^2} + \frac{1}{9}\frac{\alpha}{9^2} - \frac{1}{11}\frac{3\alpha}{9^3} + \frac{1}{13}\frac{\alpha}{9^3}$, &c.

Theref. *the* (*Radius is to* ½ *Periphery, or*) *Diameter is to the Periphery, as* 1,000, &c. *to* 3.14159265358979323 84.62643383279.5028841971.6939937510.5820974944. 5923078164. 4062862089. 9862803482. 5342117067. 9 +, True to above a 100 Places; as Computed by the Accurate and Ready Pen of the Truly Ingenious Mr. *John Machin*: Purely as an Instance of the Vast advantage *Arithmetical Calculations* receive from the *Modern Analysis*, in a Subject that has bin of so Engaging a Nature, as to have employ'd the Minds of the most Eminent Mathematicians, in all Ages, to the Consideration of it. For as the exact Proportion between the *Diameter* and the *Circumference* can never be exprest'd in Numbers; so the Improvements of those Enquirers the more plainly appear'd, by how much the more Easie and Ready, they render'd the Way to find a Proportion the nearest possible: But the Method of Series (as improv'd by Mr. *Newton*, and Mr. *Halley*) performs this with great Facility; when compared with the Intricate and Prolix Ways of *Archimedes*, *Vieta*, *Van Ceulen*, *Metius*, *Snellius*, *Lansbergius*, &c. Tho' some of them were said to have (in this Case) set Bounds to Human Improvements, and to have left no-

nothing for Posterity to boast of; But we see no reason why the indefatigable Labor of our Ancestors should restrain us to those Limits, which by means of the *Modern Geometry*, are made so easie to surpass.

Variety of other *Series* might be given for this purpose, tho' probably none so simple as this.

39. If $r = 1.000$, &c. the Length of the Arc of 1′ will be $(3,14159$, &c. $\div 180 \times 60)$, 00029.08882.08665.72159.61539.48461, &c. $(=\lambda)$; Let n be the Number of Minutes in any other Arc a, then will $a = n\lambda$: So that, by the foregoing Rules and Series, the *Natural Sine*, *Cosine*, *Tangent*, &c. of any Arc is readily made to any desired Number of Places; as also their *Logarithms*, and that from the Arc it self.

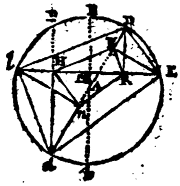
Other *Theorems* might have bin added for the Calculating of a *Table* of such Numbers; but these alone were thought sufficient to perform the same, by various easie Ways.

66. Let BL, PB, be any two Arcs (A, a) Then

ML $=$ s, A; MR $=$ s, a.

LH $=$ lR $=$ s, A $+$ s, a;

lH $=$ LR $=$ s, A $-$ s, a.



Dn $=$ v, $\overline{A+a}$; DK $=$ v, $\overline{A-a}$.

dn $=$ v, $\overline{A+a}$; dK $=$ v, $\overline{A-a}$.

Dl $=$ c, $\overline{A+a}$ $=$ 2s, $\frac{1}{2}\overline{A+a}$;

DL $=$ c, $\overline{A-a}$ $=$ 2s, $\frac{1}{2}\overline{A-a}$.

ln $=$ s, $\overline{A+a}$; LK $=$ s, $\overline{A-a}$.

dH $=$ s, A $+$ s, a; DR $=$ s, a $-$ s, A $=$ v, A $-$ v, a.

$\frac{1}{2}$ dl $=$ s, $\frac{1}{2}\overline{A+a}$; $\frac{1}{2}$ dL $=$ s, $\frac{1}{2}\overline{A-a}$.

Here 'tis plain, the *Trapezia* LdHK, lDRn, ldHn, LDKR are similar; For each has two $=$ ∠s at the Circumf. (*viz.* one $= \frac{1}{2}\overline{A+a}$, the other $= \frac{1}{2}$ the Supplement of $\overline{A+a}$,) as also 2 ∟ △s, form'd by the respective Diagonals and Sides, which are respectively similar; theref. the Sides about the $=$ ∠s will be ∷ l, Conseq. the *Trapezia* are all similar.

By

By this Scheme, which I receiv'd from the Learned Mr. *Halley*, an endless Variety of useful *Theorems* may be deduc'd, and that by Inspection only; Such as these for Instance.

$$1.\ s,\overline{A+a} : s,A - s,a :: s,A + s,a : s,\overline{A-a}$$
$$:: v,\overline{A+a} : s,A - s,a$$
$$:: s,A - s,a : v,\overline{A-a}$$
&c.

$$2.\ s,A + s,a : s,A - s,a :: t,\tfrac{1}{2}\overline{A+a} : t,\tfrac{1}{2}\overline{A-a}$$

$$3.\ v,A - v,a : s,\tfrac{1}{2}\overline{A+a} :: s,\tfrac{1}{2}\overline{A-a} : \tfrac{1}{2}\,r,$$

$$4.\ s,\tfrac{1}{2}\overline{A+a} : s,A + s,a :: s,\tfrac{1}{2}\overline{A-a} : s,\overline{A-a}$$
$$(::\ r\ : 2s,\tfrac{1}{2}\overline{A-a})$$

Theref. in *Equidif*. Arcs, (where m is the Mean between a & e, its Equidist. Extremes) 'tis,

$$s,m : s,a + s,e :: s,M : s,A + s,E,$$ (as in *Art*. 22, 54.)

67. The *Diameter* of a Plane Figure is a Right Line bisecting all Parallels to the Tangents. The Lines bisected are called *Ordinates* to that Diameter. The Point of Concourse of all Diameters is called the *Centre* of the Figure. The Intersection of the Curve and Diameter is called the *Vertex*.

That Diameter, to which the Ordinates stand at right Angles, is called the *Axis*. The common Intersection of the Diameter and Ordinate is called the *Point of Application*. The Segment of the Diameter intercepted between the Vertex and the Point of Application is called the *Abscissa* or *Intercepted Diameter*.

And *Geometric Lines* are distinguish'd according to the Dimensions of the *Equation* expressing the Relation between the *Ordinates* and *Abscissæ*; or according to the Number of Points in which they may be cut by a right Line. Then a *Line* of the *First Order*, will be a *Right Line*; Those of the *Second*, or *Quadratic Order*, will be *Conic Sections*; and those of the *Third*, or *Cubic Order*, will be the *Cubic Paraboloid*, &c. Therefore a *Curve of the* 1st, 2d, 3d, &c. *Kind*, is the same with a *Line* of the 2d, 3d, 4th, &c. *Order*.

And

And a *Line* of the *Infinitesimal Order*, is that which a right Line may cut in an Infinite Numbers of Points; as the *Spiral*, *Cycloid*, &c. and all those generated by Infinite Revolutions of a Radius.

68. Of Curves of the *First Kind*.

If in any Right Line, upon a Plane, there be taken two Points φ, f; *an Infinite Number of Points* (p) *may be found, so that* fp ± φp = vu *or* fl *a given Right Line, and* pφ = pl *every where. For with* vu *(or* fl*) from* f, *describe the Arc* gl, *draw the Lines* fl, *make the* ∠ lφp = ∠ φlp, *or* pl = pφ; *And the Curve drawn thro'* p,

1. *If it be* fp ± φp, *is called an* Ellipse, *or* Hyperbola; *as in* Fig. 1. & 2.

2. *If* f *be infinitely distant from* φ, *the Arc* gl *will become a Right Line perpendicular to* vu, *and the Curve is called a* Parabola.

3. *If* f *and* φ *coincide, the Curve will be a* Circle, *whose Radius is* ½fp + ½φp.

Therefore, If upon a Plane, any Right Line vu *be taken, and therein* vφ = uf, *and the Points* m *within, or without* φ *and* f; *and Arcs described with* vm, um, *from* φ, f, *will intersect in the Curve of an* Ellipse, *or* Hyperbola.

And in any indefinite Right Line fg (Fig. 3.) *let* vg = vφ, *and* pa *Perp.* fg; *with* ga, *from* φ *cut* pa *in* p, π, *the Curve Line drawn thro' them will be a* Parabola.

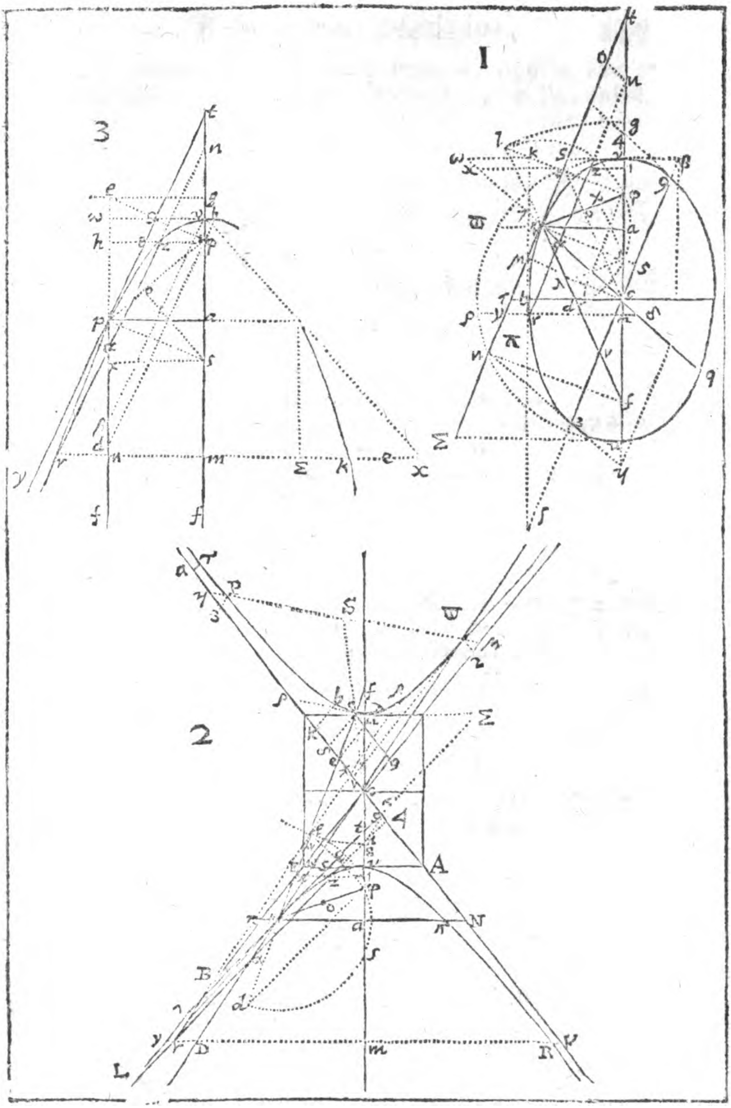
The Point *v* is the *Vertex*, pa (*y*) an *Ordinate*, va (*x*) the *Abscissa*, φ, f the *Foci*, vu (*t*) the *Transverse Axe*.

'Tis evident the *Parabola* has but one *Focus* φ, so that *t* is Infinite. Therefore the Properties of the *Parabolic* Curve are easily drawn from those of the other; yet to gratifie the Learner, we chose to do them separately.

In the Ellipse *and* Hyperbola.

1. *Thro'* p *draw* pt *making* ∠ lpt = ∠ tpφ; *Then shall* pt *be a Tangent to Curve in* p. *For, making* pl = pφ, *and taking any Point* ε *in the Line* pt, *then* εφ = εl, *but* fl (vu) < fε + εl (*in the Ellipse*), *and* fl (vu) + εl > fε (*in the Hyperbola*). *Th.* ε *is without the Curve.*

2. *Draw*



Place this Figure at Page 246.

2. *Draw* cε || fp, *cutting* tγ *in* ε, *and* φp *in* o, *Then* ∠ φεp (∠lpε) = ∠opε. *And* εo = po = (*bec.* φc = ½φf, *and th.* co = ½fp) ½pε, *Th.* cε (½φp ∽ ½fp) = cv. *Th.* (*bec.* φ,ε,p, *are in a Semicirc. whose Centre is* o) *the* ∠ φεp = ∟.

3. *Let* cv^2 ($\frac{1}{4}t^2$) : vφu ($\overline{t \mp q} \times q$) : : vu (*t*) : *p* = *Parameter*; *Then* (*bec.* $\frac{1}{4}t^2$ ($\frac{1}{4}t \times t$) : $\frac{1}{4}tp$ ($\frac{1}{4}t \times p$) : : $\frac{1}{4}t^2$: $\overline{t \mp q} \times q$) $\frac{1}{4}tp = \overline{t \mp q} \times q$, or vφu.

4. *Draw* c9 || tp, *cutting* pf *in* v, *Then* (*bec.* cv || pε, *and* cε || fp) pv (cε) = cv, (by 2.) *Supply the Hyperb. Figure.*

5. *From* p, *thro'* φ *describe a Circumf. cutting* vu *in* ς, *and* fp *in* d, l; *Bisect* fd *in* v; *then*
½fl : ½fφ : : ½fς : ½fd, *i. e.* cv : cf : : ca : fv, *th.*
1. cv : cv + cf : : ca : ca + fv : : cv + ca : cv + cf + ca + fv.
[*that is*, cv : fv : : ua : fp + fa,] And
2. cv : cv ∽ cf : : ca : ca ∽ fv : : cv ∽ ca : cv − cf − ca + fv
[*that is*, cv : fu : : va : fp − fa;]
Th. cv^2 : vfu : : vau : ($fp^2 - fa^2$) ap^2.

i. e. $\frac{1}{4}t^2 : \overline{t \mp q} \times q : : \overline{t \mp x} \times x : y^2 : : t : p : : \overline{t \mp X} \times X : Y^2$

6. *And* (*bec.* ∠ lpt = ½∠ lpφ = ∠ pdφ, *th.* tp || φd || cv.) fv : fc : : cε : ct. *but* (*by* 5) fv : fc : : ca : cv : : (cv) cε : ct, *that is*, $\frac{1}{2}t \mp x : \frac{1}{2}t : : \frac{1}{2}t : \frac{1}{2}t \pm a$ (which is also true, if *t* be the Diameter) : : ($\frac{1}{2}t + \frac{1}{2}t \mp x : \frac{1}{2}t + \frac{1}{2}t \pm a$: :) $t \mp x : t \pm a$
: : ($\frac{1}{2}t$ ∽ $\frac{1}{2}t \mp x : \frac{1}{2}t$ ∽ $\frac{1}{2}t \pm a$: :) $x : a$

7. Theref. $t \pm a : a : : t \mp x : x$. And,

8. $\frac{1}{2}t \mp x : x : :$ ($\frac{1}{2}t : a : : \frac{1}{2}t + \frac{1}{2}t \mp x$ or) $t \mp x : x + a$

9. Or, $\frac{1}{2}t \mp x : t \mp x : : \frac{1}{2}t : t \pm a : : x : x + a$. And,

10. $t \pm a : \frac{1}{2}t \pm a : : x + a : a$. *Th.* uΣ : cτ : : ap : vs

11. Also, $t : p : :$ ($\overline{t \mp x} \times x$) $\overline{\frac{1}{2}t \mp x} \times \overline{x + a} : y^2$

12. Th. $\frac{y}{x + a} = \frac{\frac{1}{2}t \mp x}{y} \times \frac{p}{t}$, or $\frac{ap}{at} = \frac{ca}{ap} \times \frac{p}{t}$

13. *If* vs, uΣ (|| ap) *meet* tγ *in* s, Σ; *then*
{ vs : ap : : (vt : at : :) cv : ua (*by* 8.)
{ uΣ : ap : : (ut : at : :) cv : va (*by* 9.) } *th.*
vs × uΣ : cv^2 (: : ap^2 : vau : : *p* : *t*) : : $\frac{1}{4}tp$: $\frac{1}{4}t^2 = cv^2$.

Th.

Th. $vs \times u\Sigma = \frac{1}{4}tp = \overline{t+q} \times q = c\tau \times ap$.

Note, in the *Hyperbola*, ſupply the Fig. becauſe here omitted to avoid Confuſion; And obſerve the ſame in other places where 'tis required.

14. *If from the Interſect.* ϵ, n *of any Tangent* $t\gamma$, *and a Circle deſcr. on* vu, *Perpendiculars* $\epsilon\phi$, nf *be erected; Then* (*bec.* $\triangle$s tvs, $tu\Sigma$, *Sim.* $\triangle$s $tn\phi$, tnf,) $u\Sigma : nf :: \epsilon\phi : vs$, *Th.* $vs \times u\Sigma = \epsilon\phi \times nf = (\overline{v\phi \times \phi u}|^{\frac{1}{2}} \times \overline{vf \times fu}|^{\frac{1}{2}})\ v\phi \times \phi u$ or $vf \times fu = \overline{t \mp q} \times q$ ($= \frac{1}{4}tp = c\tau \times ap$) *Th.* ϕ *and* f *are the Foci.*

15. *Alſo drawing* $p\varsigma \perp pt$, *or* $\parallel \phi\epsilon$, *and* $\varsigma\chi$, $\varsigma\lambda \perp p\phi$, pf; *then* (*bec.* $\triangle$s $fp\varsigma$, fpn, *Sim.* $\triangle$s $fl\phi$, $\varsigma p\lambda$) fl (or t) $\times$ $p\lambda = \phi l$ (or $2\epsilon\phi$) $\times fn = \frac{1}{2}p \times t$; *Th.* $p\lambda = p\chi = \frac{1}{2}p$.

16. *Let* vs, $u\Sigma$ (*Perp.* vu) *interſect any Tangent* $\tau\tau$ *in* s, Σ, *draw* sf, Σf: *Then* (*bec.* $vs \times u\Sigma = uf \times fv$ (*by* 14) *th.* $u\Sigma : uf :: fv : vs$, *and the Included* $\angle$s Σ uf, fvs *are* $\llcorner$, *th.* $\triangle$s *are Sim. and* $\angle f\Sigma u = \angle vfs$, *but* $\angle f\Sigma u + \angle uf\Sigma = \llcorner = \angle vfs + \angle uf\Sigma$, *th.*) $\angle sf\Sigma = \llcorner$. And $s\tau = \phi\tau = f\tau = \Sigma\tau$. *Th. if* $s\Sigma$ *be made the Diameter of a Circle, the Circumf. will paſs thro' the Foci* f, ϕ. (*Supply the Figures.*)

17. *Draw* zr $\parallel$ tp, *cutting* cp *in* α; *Then* $z\alpha = \alpha r$.

For (*producing* rz *to* n, *and* cp *to* δ, x, ω.) $\frac{\iota z}{\iota n}$ ($= \frac{ap}{at} = \frac{ca}{ap} \times \frac{p}{t}$) $= \frac{cv}{v\omega} \times \frac{p}{t}$, *th.* $cv \times n\iota \times z\iota = (v\omega \times z\iota^2 \times \frac{t}{p} =)\ v\omega \times v\iota u$ (*by* 5,) *th.* $\overline{cv} \times \overline{\iota n} \times \overline{\iota z} \overset{+}{\underset{\sim}{}} v\omega \times c\iota^2$ ($= \overline{v\omega \times v\iota u} \overset{+}{\underset{\sim}{}} \overline{v\omega \times c\iota^2}$ or $\overline{\overline{t \mp x} \times x \pm \overline{\tfrac{1}{2}t \mp x}|^2} \times v\omega) = v\omega \times cv^2$, *th.* $v\omega \times cv = c\iota \times (\frac{c\iota \times v\omega}{cv})\ \iota x \overset{+}{\underset{\sim}{}} \iota n \times \iota z$, *th.* $\triangle cv\omega = \triangle c\iota x \overset{+}{\underset{\sim}{}} n\iota z =$ (*by the like arguing*) $\triangle cm\delta \overset{+}{\underset{\sim}{}} \triangle nmr$, *th.* $\triangle \alpha zx =$ *and Sim.* $\triangle \alpha r\delta$, *Th.* $z\alpha = \alpha r$; *which therefore are Ordinates to the Diameter* pq.

Therefore all Lines paſſing thro' the Centre are Diameters; And all Diameters paſs thro' the Centre; And all Ordinates are parallel to the Tangent at the Vertex of their Diameters.

18. *Hence*

18. *Hence* (*bec.* $v\omega : ap$ (: : $vc : ac$) : : $tc : vc$ (*by* 6) $\triangle\, cpt$ ($\triangle\, cap \pm\!\!\backsim tap$) $= \triangle\, cv\omega$, *th.* $\triangle\, tvs = \triangle\, sp\omega$; *And Fig.* $v\iota x\omega = \triangle\, \iota nz$, *also Fig.* $\alpha ptn = \triangle\, \alpha xz$, *i. e.* $\overline{\alpha p \times \alpha n + pt} \times \alpha p = \alpha x \times \alpha z = \alpha p\ (\sigma n) \times \overline{\alpha n + pt}$.

19. *And* $p\omega : ps$ (: : $\alpha x : \alpha z$) : : $2pt : P =$ *Parameter to the Diameter* pq. *For* $P \times \frac{p\alpha q}{pq} \left(= \frac{\alpha z \times 2pt}{\alpha x} \times \frac{p\alpha q}{pq} = \frac{\alpha z}{\alpha x} \times p\alpha q \times \left(\frac{2pt}{pq} \text{ or } \frac{pt}{pc} \text{ or } \frac{\sigma t}{\sigma n} \right) \frac{\sigma t}{pa} = \frac{\alpha z}{\alpha x} \times \left(\alpha q \times \sigma t, \text{ or } \overline{\alpha c + cp} \times \sigma t, \text{ or } \overline{tp + n\alpha} \times (\sigma n)\ \alpha p, \text{ or } \right) \alpha x \times \alpha z = \frac{\alpha z}{\alpha x} \times \alpha z \times \alpha x \right) = \alpha z^2 = \alpha r^2$.

Theref. the Squares of the Ordinates to any Diameter (D) *are as the Rectangles under the Segments of that Diameter,* i. e. $\overline{D \mp x} \times x : : y^2$ (: : $D : P$) : : $\overline{D \mp X} \times X . r^2$

And (*bec.* $\frac{1}{4}d^2 : \frac{1}{4}c^2 : : d : p$) c (*the Conjug. Diam.*) *is* $= \sqrt{dp}$.

20. *Let* $c\varphi = r$, $v\varphi = q$, $\varphi f = \delta$, *then*

If $x = q$, ($t : p : : \frac{1}{4}tp$ ($\overline{t \mp q} \times q$) $: y^2$) $y = \frac{1}{2}p$

And $t : 2y : : \overline{t \mp q} \times q : : y^2$, Th. $y = \overline{t \mp q} \times q \div \frac{1}{2}t$

21. Since $\frac{1}{4}t^2 : (\frac{1}{4}pt = \overline{t \mp q} \times q =) \ \overline{\frac{1}{2}t \mp q}|^2 \backsim \frac{1}{4}t^2 : : \frac{1}{2}t : \frac{1}{2}p$. Th. $\frac{1}{4}t^2 : \overline{\frac{1}{2}t \mp q}|^2 : : \frac{1}{2}t : \frac{1}{2}t \mp \frac{1}{2}p$, i. e. $\frac{1}{2}t : r : : 1 : \frac{1}{2}t \mp \frac{1}{2}p$. Th. $\frac{1}{2}t : r : : r : \frac{1}{2}t \mp \frac{1}{2}p$. Th. $r = \overline{\frac{1}{4}t^2 \mp \frac{1}{4}tp}|^{\frac{1}{2}}$; $q = \overline{\frac{1}{4}t^2 \mp \frac{1}{4}tp}|^{\frac{1}{2}} \backsim \frac{1}{2}t$; $\delta = \overline{t^2 \mp tp}|^{\frac{1}{2}}$

22. *Sup. any Ordinate* mr, *of an Ellipse, to meet the Focal Tangent in* γ; *Then* $\varphi r = m\gamma$. *Let* $c\varphi = r$, $cv = b$, $cm = n$, *then* φp ($\frac{v\varphi u}{cv}$) $= \frac{b^2 - r^2}{b} = \frac{1}{2}p$, ct ($\frac{cv^2}{c\varphi}$) $= \frac{b^2}{r}$

th. $\varphi t = \frac{b^2 - r^2}{r}$, *and* $mt = \frac{b^2 \pm rn}{r}$, *th.*($b : r :: \frac{b^2 \pm rn}{r} :$) $\frac{b^2 \pm rn}{b} = m\gamma$; *But* (*bec.* $cv : \frac{1}{2}p :: vmu : mr^2$, *i. e.* $b : \frac{b^2 - r^2}{b} :: b^2 - n^2 :$) $mr^2 = \frac{b^4 - b^2n^2 - r^2b^2 + r^2n^2}{b^2}$, *and* $\varphi m^2 = r^2 - 2rn + n^2$, *th.* $\varphi r^2 = \frac{b^4 + r^2n^2 \pm 2rnb^2}{b^2}$; *and* $\varphi r = \frac{b^2 \pm rn}{b} = m\gamma$; *'Tis the same in the* Hyperbola *and* Parabola.

23. *Theref. if* cb *meet the Focal Tangent in* τ, *then* $c\tau$ (φb) $= cv = cu$.

24. (*Bec.* $\frac{cv^2}{c\varphi}$ (ct) : cv ($c\tau$) :: $\frac{cv^2}{c\varphi} \pm cv$ ($\frac{ut}{vt}$): $cv \pm c\varphi =$) $u\Sigma = u\varphi$, *or* $= vs = v\varphi$.

25. *The same being still supposed*, (*that is*, *that* $x = q$) *draw* pu *cutting any Ordinate* mr *in* θ; *Then* $\gamma\theta = \varphi m$. (Supply the *Figure.*)

26. *If on* vu, *a Circle be described*, *then bec.*
$ap^2 : mr^2 :$ (:: vau : umv) :: $a\varpi^2$, $m\varrho^2$, *th.*
$ap : mr :: a\varpi : m\varrho$; $ap : p\varpi :: mr : r\varrho$,
Or $ap : a\varpi$ (:: $mr : m\varrho$ :: cb : cv) :: $c : t$.
Th. all (ap's : *all* $a\varpi$'s ::) *Ellipse* : ⊙ t :: $c : t$ :: $ct : t^2$ :: ⊙ $\sqrt{tc}$: (⊙ $\sqrt{t^2}$) ⊙ t. *Th. Ellipse* = ⊙ $\sqrt{tc}$. *And the Ellipses* $E : e :: TC : tc ::$ (bec. $c = \sqrt{tp}$) $P^{\frac{1}{2}}T^{\frac{3}{2}} : p^{\frac{1}{2}}t^{\frac{3}{2}}$

27. *Parallelograms circumscribing an Ellipse*, *having their Sides parallel to the Conjugate Diameters*, *are equal.* *For since* $\Delta\mu p7 = \Delta\mu\tau b$ (*by* 18) *and* ∠ *at* $\mu =$, *Th.* $\tau\mu : \mu p :: \mu 7 : b\mu$, *th.* $\tau p : \mu p :: b7 : \mu 7$) $\Delta c\tau p : \Delta c\mu p :: \Delta c7b : \Delta c\mu b$, *but* $\Delta c\tau p = \Delta c7b$ (*by* 18) *th.* $\Delta c\mu p = \Delta c\mu b$, *and* $\Delta fc\mu$ ($\frac{1}{2}$ ▭ ft) $= \Delta tc\mu$, *th.* $\Delta cbf = \Delta cpt$, *th.* ▭ cf = ▭ ct; *Also* ▭ cy ($2\Delta c3y = 2\Delta c94 = 2\Delta cv\beta$) = ▭ βc, *but*
{ ▭ yc : ▭ πc (:: y3 : 3π :: πp : pt) :: ▭ πc : ▭ ct
{ ▭ βc : ▭ kc (:: βv : vk :: kb : bf) :: ▭ kc : ▭ cf.

Th.

Th. $\square\pi c = \square kc$, *and theref.* $4\square\pi c = 4\square kc$.

28. *Ib. if* cp, c9 *be Semi-conjugate Diameters, and* p5 $\perp$ c9, *then* $c9 \times p5$ ($\square$ p9) $= cv \times cb$.

Of the Hyperbolic Asymptotes.

29. *Since* $ct \times ca = cv^2$ (*by* 6) *if* $ct = o$, *then* $ca = \infty$; $vt = ut$, *th.* $vs = u\Sigma = \overline{vs \times u\Sigma}|^{\frac{1}{2}} = \overline{\tfrac{1}{4}tp}|^{\frac{1}{2}}$; *Th. if* vE *or* vA ($\perp$ vu) *be made* $= \overline{\tfrac{1}{4}tp}|^{\frac{1}{2}}$, *then* cE, cA *are called Asymptotes to the Hyperbola or Opposite Hyperbolas.*

30. $ca^2 : an^2 :: (cv^2 : vE^2 :: \tfrac{1}{4}t^2 : \tfrac{1}{4}tp :: t : p ::) vau : ap^2 :: ca^2 - vau : an^2 - ap^2$; *but* $ca^2 > vau$, *th.* $an^2 > ap^2$; *Th.* n *will always be without the Hyperbola, and conseq.* cn, *tho' infinitely continued, shall never meet the Curve.*

31. *Also* $cv^2 = ca^2 - vau$, *th.* $vE^2 (an^2 - ap^2) = npN$ *or* $n\pi N$; *And* $vE^2 = \gamma rK$ *or* γRK, *th.* $npN = \gamma rK$, *but* $pN < rK$, *th.* $pn > \gamma r$, *Th. the Hyperbola and its Asymptotes do continually approach nearer.*

32. *And if* cL, cK *are Asymptotes, and by any point* (p) *of the Curve, a Tangent be drawn cutting* cL, cK *in* 7, 5, *Then* $p5 = \overline{\tfrac{1}{4}p\varpi \times P}|^{\frac{1}{2}} = p7$; $Lr \times r4 = 7p \times p5 = \tfrac{1}{4}p\varpi \times P$; *And* $Lr = z4$.

33. *Draw* r3, *parallel to any Semi-diameter* pc, *cutting the Asymptotes in* B, χ; *Then* $Br\chi = cp^2$: *For the Ordinates applied to the points* p, α, *meet the Asymptotes in* 7, 5, *and* L, 4, *th.* $\frac{7p}{pc} \times \frac{p5}{pc} = \frac{Lr}{Br} \times \frac{r4}{r\chi}$; *Th.* &c.

And (*bec.* $Br\chi = cp^2 = c\varpi^2 = \chi 3B$, *also* $Br = \beta\chi$) $r\chi = B\beta$.

34. *If thro' any points* P, ϖ, *of the Hyperbola, be drawn right Lines* P3, ϖ2, *parallel to the Asymptotes, Then* (*bec.* $\Delta\varpi\mu 2$. *Sim.* $\Delta Py3$, *and* $\varpi\mu = Py$, *th.* $\varpi 2 = y3$, *and* $\mu 2 = P3$,) $P3 : \varpi 2 :: (\mu 2 : y3 :: \mu c : cy :: c\mu - P3 : cy - \varpi 2 ::) c2 : c3$, *Th.* $P3 \times 3c = \varpi 2 \times 2c$.

35. *If the Segments* cv, cδ, c3, *of the Asymptote* cy *be* $\div$, *Then shall* vϖ, δs, 3P (*drawn* $\parallel$ *to the other Asymptote* cμ,) *be* $\div$.

For $\nu\varpi : \delta s :: (c\delta : c\nu :: c3 : c\delta ::) \delta s : 3P$.

But if $c\nu = \nu e = e\delta = \delta x$, &c. *that is*
if $c\nu$, ce, $c\delta$, ck, &c. *be in a Contin. Arith.* } *Proport.*
Then $\nu\varpi$, $e\lambda$, δs, kb, &c. *are in a Contin. Harmon.* }
And $e\lambda = \frac{1}{2}\nu\varpi$, $\delta s = \frac{1}{3}\nu\varpi$, $kb = \frac{1}{4}\nu\varpi$, &c. *as 'tis easie to demonstrate.*

Let $\varrho s \parallel y\mu$, *then* $s\delta : \varrho s :: P3 : yP$, *and* $s\delta : \varrho s :: \nu\varpi : y\varpi$. *th.* $s\delta^2 : \varrho s^2 :: P3 \times \nu\varpi : yP \times y\varpi$, *but* $s\delta^2 = P3 \times \nu\varpi$ (*by* 35) *th.* $\varrho s^2 = Py\varpi$ *or* $yP\mu$, *th.* ϱs *is a Tangent, and* $P\varpi$ *an Ordinate to the Diameter* cS, *th.* $\Delta ySc = \Delta\mu Sc$, *and Space* $SsP = Ss\varpi$, *but* $\Delta cs\delta = \Delta cs9$, *th. Space* $s\delta 3P = s92\varpi$ *but* $\square\, c\varpi = \square\, cs$, *th.* $\square\, \varpi 9 = \square\, s\nu$. *Th. Space,* $s\delta 3P = s\delta\nu\varpi$.

Likewise, if ck, $c\delta$, ce $\div$, *draw* kb, $e\lambda \parallel c\mu$, *then Space* $e\lambda s\delta = \delta skb$, *th.* $bk\nu\varpi = e\lambda P3$, *and bec.* $\nu\varpi \times P3 = \delta s^2 = e\lambda \times kb$, *th.* $\frac{P3}{e\lambda} = \frac{kb}{\nu\varpi}$, *Th. Asymptotic Spaces are as the Logarithms. So the Space* $\alpha\tau P3$ *is the Log. of the Ratio's* $\frac{P3}{\tau\alpha} = \frac{c\alpha}{c3}$; *And the Log. of the Ratio of Equality* $\frac{\alpha\tau}{\alpha\tau}$ $(= 1)$ *is* 0.

In the *Parabola*.

36. *Let* $\nu\varphi = q$, *then* $ag\ (\varphi P) = x + q$, $\varphi a = x - q$ *and* $y^2\ (p\varphi^2 - a\varphi^2) = \overline{x+q}|^2 - \overline{x-q}|^2) = 4qx = px$
And ($4q$) p *is called the* Parameter *of the Axe.*

37. *Th.* $\nu\varphi = \frac{1}{4}p$; *And* $z\varphi = 2\varphi\nu = \frac{1}{2}p$.

38. *Also* $y^2 : Y^2\ (:: px : pX) :: x : X$.

39. *Since* $x < X$, *th.* $y < Y$, *th. the Curve of a* Parabola *runs off infinitely.*

40. $p \times \overline{X - x}\ (pX - px) = \overline{Y+y} \times \overline{Y-y}\ (Y^2 - y^2)$ *that is,* $p \times \Sigma\pi = r\Sigma \times \Sigma k$.

41. *Since* $y^2 : Y^2\ (:: x : X) :: y : Y + e$
Th. $\overline{Y+e} \times y\ (= Y^2) = Xp$.

42. *Any*

42. *Any right Line* nz *cutting the Axe and Curve, will again if produc'd, cut the Curve in* r, *so that* vι, vn, vm ∺ ; *Suppose it so, let* rm, zι ⊥ nm, *then* vι : vn :: vn : vm :: vι + vn : vn + vm :: nι : nm, *and* $\iota z^2 : mr^2 :: n\iota^2 : nm^2 :: v\iota^2 : vn^2 :: v\iota : vm$, *th. the Point* r *is in the Curve.*

43. *Draw* pt, *so that* tv = va, *then* pt *shall touch the Curve in* p ; *since any other point* (ε) *in* pt *will be without the Curve: For (drawing* ει || pa) $\frac{\iota\varepsilon^2}{ap^2} = (\frac{\iota t^2}{at^2} = \frac{\frac{1}{4}\iota t^2}{va^2} > \frac{\iota vt}{va^2} = \frac{\iota va}{va^2} = \frac{\iota v}{va} =) \frac{\iota z^2}{ap^2}$ *Th.* ιε > ιz.

44. *Th.* φp (ga) = tφ ; *and if* dp || fv , *then* ∠dpγ (∠φtp) = ∠ φpt.

45. *If* pς ⊥ pt, *then* aς = ½*p*; *for* (*p*×av) ½*p*×2av = (αp²) 2av×aς.

Th. if ςϰ, ςo ⊥ pd, pφ, *then,* (*bec.* Δs ςpϰ, ςpo, ςpa *are* = *and Sim.*) pϰ = po = ςa = ½*p*.

46. *And since* φp = φt, *and* ∠ ςpt = ∟, *the Points* ς, p, t, *are in a Semi-Circ. whose Centre is* φ, *Th.* φp = φς = φt.

Hence the ∠ *made by the Tangents, from the Extremities of any Ordinate, is* ½ *that at the Focus made by Lines from the same Extremities.*

Th. if the Ordinate passes thro' the Focus, the ∠ *made by the Intersection of its Tangents will be a Right one.*

And (*bec.* av = vt) vs² (¼2p²) = ¼ av×*p*.

47. *Let any Rt. Line* zr, *drawn* || tp, *cut the Curve in* z, r, *and* lpd (|| va) *in* α ; *let* rm, zι (|| pa) *cut* pα *in* n, h. *Then* rα = zα.

For Δ apt (□ωa, *bec.* 2av = at) : Διzn (::)ap² : ιz² :: av : ιv) :: □ωa : □ωι, *th.* Διzn = □αι ; *and* (*by the like arguing*) Δmrn = □ωm, *th. Fig.* ιzrm (Δrnm — Δzrι = □ωm — □ωι) = □hm, *th.* Δrαn = *and Sim.* Δzαh *Th.* rα = zα, *and* pd *is a Diameter, whose Ordinate is* rz.

Hence, in the Parabola, every Line parallel to the Axe is a Diameter. And all Diameters are parallel the one to the other. Also all Ordinates are parallel to the Tangent at the Vertex of their Diameter. Theref. a Line, thro' the Vertex of the Diameter, drawn parallel to the Ordinate is a Tangent to the Parabola.

48. *And*

48. *And Fig.* $_n$ptm (Δpta+$\square$mp = aω+mp=$\square$mω) = Δmrn, *th.* $\square\alpha$t= $\Delta\alpha$rn, i: e. 2pt×pα = αr×αn.

49. *Let* pω : ps : : 2pt : *P*, *i* : *e.* va : pt : : pt : *P* = *Parameter to the Diameter* pd; *For* (2pt×pα× ($\frac{\alpha r}{\alpha n}$ *or* $\frac{ps}{p\omega}$ *or*) $\frac{P}{2pt}$ =) pα×*P* (= $\overline{\alpha r \times \alpha n} \times \frac{\alpha r}{\alpha n}$) = αr^2 = rz^2. *Th. the Abscissæ of every Diameter are as the Squares of their respective Ordinates.*

50. *Since* ($pt^2 = v\beta^2$ =) $4x^2+y^2 = Px$, *Th.* P = ($4x+\frac{y^2}{x}$ =) $4x+p$ (4*q*) *And* $P-p=4x$.

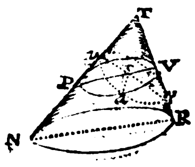
Hence p *is the least of all the Parameters.*
Also $\frac{1}{4}P(x+q)$ =ag *or* φp. *And if* φs *be* $\perp$ *pt, then* $s\varphi^2$ = ($p\varphi^2-\frac{1}{4}pt^2(ps^2)$ = $\overline{x+q}^2 - x^2 - xq$ = $\overline{x+q}\times q$ =) $p\varphi \times q$; *Th.* $s\varphi^2$ *is ever as* pφ.

51. *If an Ordinate* db *to the Diameter* dp *passes thro' the Focus* φ, *then* $\frac{1}{4}P$ (*x*+*q* *or* vt+vφ *by* 9) = dp. *Also* (*bec.* *P*×dp = $\frac{1}{4}P^2$ = db^2) $\frac{1}{2}P$ = db = 2dp (dp+pφ) = ts.

These are the chief Properties of *Curves of the First Kind*; tho' there may be an innumerable Variety of others, most of which, if thought worth the Labour, may be drawn from the foregoing Principles. And such whose Curiosity prompts them to a farther Inquiry into these Matters, may have Recourse to larger Volumes.

69. Of *the Conic Sections.*

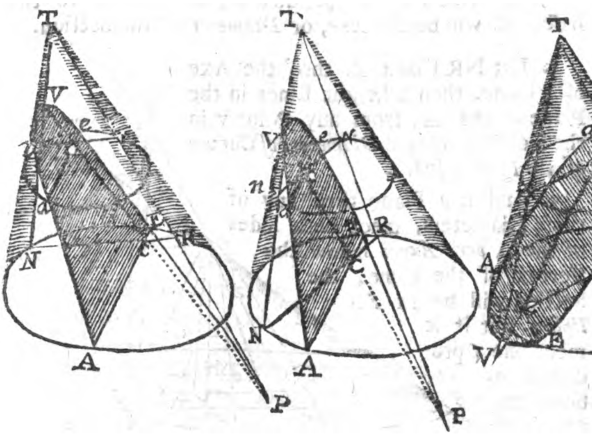
1. *If a Cone be cut by a Plane thro' the Axe, the Section will be a* Triangle. *But if cut* Parallel, *or* Subcontrary *to the Base, the Section will be a* Circle: For a Cone may be consider'd as compossed of an infinite Number of Circles, all parallel to the Base; And the *Subcontrary* Section is evidently Similar to the Base, *Bec.* ΔpcV



Sim. ΔPcv, *th.* pc×cv=(Pc×cV=) ca^2, *th.* vap *is a Circle.*

2. *If in any Section of a Cone, a right Line* (VP) *terminating in the Curve intersect any two Parallel Lines* (AE, ae)

ae) *the Rectangles of the intercepted Segments of that Line will be, as the Rectangles of the Segments of the respective Parallels:* For

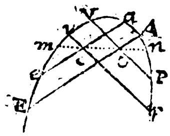


Let the right Line VP be the Common Section of the $\triangle$NTR, (passing thro' the Vertex of the Cone, but not thro' the Axe) and the Section AVE.

Then $RC \times CN = EC \times CA$, and $rc \times cn = ec \times ca$ (by 64. 23.) Also $rc : RC :: cP : CP$, and $nc : NC :: Vc : VC$.

Th. $rc \times cn : RC \times CN$ }
Or $ec \times ca : EC \times CA$ } $:: Pc \times cV : PC \times CV$.

3. *Th. In any Conic Section, if two Parallels are cut by two others, and all terminate at the Curve, the Rectangles of the Segments shall be Proportional,* i.e. $ace : ACE :: (mcn : mCn ::) pcv : PCV$.

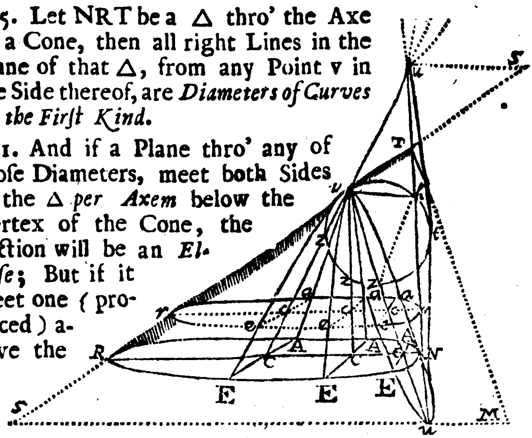


4. Let any Plane AvE cut the Base of a Cone in A, E, then NR drawn Perp. to AE, and thro' the Centre, bisects AE: Let a Plane NTR cut the Cone thro' NR and the *Axe*; and a Plane (nare) cut it || to the Base, then the com. Sect. ae, and nr, of the Planes nar, ave, and nar, nTr are || to EA and NR; but nr passes thro' the Centre, th. bisects ea in c; also vC (the com.

com. Sect. of the Plane AvE and that *per Axem*) bisects all right Lines drawn || to AE in the Section, th. vc is a *Diameter* of the Section, and AC, ac, are *Ordinates*. If the Plane *per Axem* be Perpendicular, or Oblique to the Base, vC will be the *Axe*, or *Diameter* of the Section.

5. Let NRT be a △ thro' the Axe of a Cone, then all right Lines in the Plane of that △, from any Point v in the Side thereof, are *Diameters of Curves of the First Kind*.

1. And if a Plane thro' any of those Diameters, meet both Sides of the △ *per Axem* below the Vertex of the Cone, the Section will be an *Ellipse*; But if it meet one (produced) above the



Vertex, the Section will be an *Hyperbola*, and the Sections of the opposite Cones will be equal *Hyperbolas*. For

1. vc : vC :: rc : RC, *and* uc : uC :: nc : NC
Th. vcu : vCu :: (ncr : NCR ::) ac^2 : AC^2

2. *Draw* TM, vh || vu, NR; *make* vu : vh :: us : P *the Parameter*; *For* vc : cr :: vu : vs :: TM : Ms, *and* uc : cn :: vu : vh :: TM : Mu, *th.* vcu : (ncr) ca^2 :: (vu^2 : us×vh (vu×P) ::) vu : P. *Also* TM^2 : sMu :: vu : *P*.

The same appears more general by 69, 2. For (if *D*, *d*, be the *Transverse* and *Conjugate Diameters*; *y*, *Y*, *Ordinates* to the *Diameter D*; x, *X*, their *Abscissæ*) $x \times \overline{D \mp x} : X \times \overline{D \mp X} :: y^2 : Y^2$. And if an *Ordinate* (*Y*) be applied to the Centre, 'twill become $= \frac{1}{2}d$, then $\frac{1}{4}D^2 : \frac{1}{4}d^2 :: x \times \overline{D \mp x} : y^2$, Also, let (1) $D : d :: d : P$; And (2) $D : x :: P : z$, then (bec. $\overline{D \mp x} : x :: \overline{P \mp z} : z$ (by 2) th. $z \times \overline{D \mp x} = x \times \overline{P \mp z}$, but $D : P :: D^2 : d^2$ (by 1) :: $x \times \overline{D \mp x} : y^2$

(by)

(by 69.2) : : $x : z$ (by 2) : : $x \times \overline{D \mp x} : z \times \overline{D \mp x}) x \times \overline{P \mp z}$ $= y^2 = \overline{DPx \mp Px^2} \div D$. And bec. $DP = d^2$, $dp = D^2$, th. $Pp = Dd$, and $\frac{1}{4}d^2$ ($\frac{1}{4}DP$) $= \frac{1}{2}D \times \frac{1}{2}P$.

2. If the Plane be Parallel to one Side of the △ *per Axem*, the Section will be a *Parabola*. For

1. vc : vC : : (cr : CR : : ncr : NCR : :) $ca^2 : CA^2$.

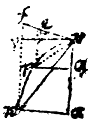
2. Make NTR : NR^2 : : Tv : P, the *Parameter*; For NC (vh) : vT : : NR : RT, and RC : Cv : : RN : NT, th. CA^2 (NCR) : TvC : : (NR^2 : NTR : : P : Tv : :) $P \times$vC : TvC, th. $CA^2 = P \times$vC. Or by 69. 2. (bec. $D - x = D - X$, since $D = \infty$) $x : X : : y^2 : Y^2$.

And if $x : y : : y : P$, then $Y^2 = PX$ every where; For $Px (y^2) : Y^2 : : (x : X : :) Px : PX$.

6. *If thro'* v *and* h *a Circle be describ'd, whose Segment* vzh *contains an* ∠ = ∠ Tvh, *the Chords* vz *shall be the* Parameters *to the* Diameters vu, vc, *&c. For in the* Ellipse *and* Hyperbola, (*bec.* △ vhz *Sim.* △ vsu *th.*) vu×P (vh×su) = vu×vz, *th.* P = vz. *And in the* Parabola (*bec.* △vhz *Sim.* △ vRC) vC×P(vh (NC) ×CR) = vC× vz, *th.* P=vz.

And ut *Tangents to the Circle from the Extremities of the Diameters, shall be the Distance of the* Foci; *For* $ut^2 = vu^2 \mp vz \times vu$ (*by* 64. 19.) = *Square of the Distance of the* Foci (*by* 68. 22.)

7. *In any Conic Section* vpπ, *where* ap, *a*π *are Ordinates*, v*a* *the Diameter*, v*v* *a Tangent*; *Let* pc, πv (∥ v*a*) *be infinitely small*, *Then* (*since* $ap^2 = \overline{p \div t} \times t \times \overline{va \mp va^2}$ = (*bec.* va : t : : 1 : ∞) $\overline{p \div t} \times t \times$va, *and so* $a\pi^2 = \overline{p \div t} \times t \times$ v*a*, th. $ap^2 : a\pi^2$: : va : v*a*, *or*) $vc^2 : vv^2$: pc : πv. *Th.* cp : cv : : (va : ap : : ap : $tp \div t$: :) 1 : ∞. *And* (*bec.* $vp^2 = (vc^2 + cp^2 =) vc^2$) vp = vc. *Th. Chord* vp = *Tangent* vc = *Arc* vp; *And* pc : πv : : $vp^2 : v\pi^2$.



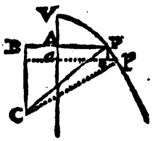
8. *Let any Line* vf *cut* pc, πv, *produced, in* e, f, *then* cp : ce : : 1 : ∞, *th* ec = ep, *and* △ vce = △ v*v*f. *Th.* △vpe : △vπf : : (△vce : △v*v*f : : $vc^2 : vv^2$: :) $ve^2 : vf^2$.

70. Curve Lines are by some distinguish'd thus; That, whose Nature is express'd by an Equation, wherein the

two Indeterminate Quantities represent Right Lines, or one of them a Curve Line, they call an *Algebraic*, or *Transcendent Curve:* And those, the Exponent of whose Equation are variable Quantities, they call *Exponential Curves.*

But since the Relations between the Indeterminate Quantities in the Equation of a Curve may be infinitely varied, theref. the Number of such Curves is infinite; Some of which, besides those of the *First* and *Second Kind*, *Geometers* have consider'd; But of these, we shall in this place, only take Notice of the Primary Properties of the most Simple *Transcendent*, and *Exponential Curve*; the *Cycloid*, and *Logarithmic Line*; their Use being very considerable.

71. If the Tangents to the Points of any Curve be produced till they become equal to their respective Parts of the Curve, their Extremities will be in another Curve; which is said to be describ'd by *Evolving* the former, the Curve Evolv'd is called the *Evoluta*, and the Tangents so prolong'd are called the *Radii of the Evoluta*, or *Radii of the Curvature*; Which therefore are all perpendicular to the Curve describ'd by the Evolution: So that the Points of the *Evoluta* are but the Intersections of Perpendiculars, to the Curve describ'd, that are infinitely near the one to the other.



Th. sup. the Ordinates AP, ap, infinitely near, and C the Intersection of the Perpendiculars to the Curve VPp, in P and p. Draw CB || AV, then Pe : Pp :: $\overline{PB}$: $\overline{PC}$, i:e $\dot{x}$: $\overline{\dot{x}^2+\dot{y}^2}|^{\frac{1}{2}}$ :: z: $\overline{\dot{x}^2+\dot{y}^2}|^{\frac{1}{2}} \times z \div \dot{x}$,

But whilst *x* and *y* increase by $\dot{x}$ and $\dot{y}$, PC becomes pC, th. does not increase, Conseq. the *Flux.* of (PC)

$$\frac{\overline{\dot{x}^2+\dot{y}^2}|^{\frac{1}{2}} \times z}{\dot{x}} \text{ is } \frac{\dot{z}\dot{x}^2+\dot{z}\dot{y}^2+z\dot{y}\ddot{y}}{\overline{\dot{x}^2+\dot{y}^2}|^{\frac{1}{2}} \times \dot{x}} = 0, \text{th. (bec.}$$

$\dot{y}=\dot{z}$) $z=\dfrac{\dot{x}^2+\dot{y}^2}{-\ddot{y}}$, but Pe : Pp :: PB : PC, i:e $\dot{x}$:

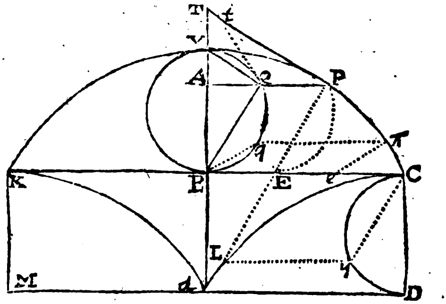
$\overline{\dot{x}^2+\dot{y}^2}|^{\frac{1}{2}}$

$\overline{\dot{x}^2+\dot{y}^2}|^{\frac{1}{2}} :: \frac{\dot{x}^2+\dot{y}^2}{-\ddot{y}} : \frac{\overline{\dot{x}^2+\dot{y}^2}|^{\frac{3}{2}}}{-\dot{x}\ddot{y}}$ the *Ray of the Evoluta* or *Radius of the Concavity* of any Curve, or the Radius of a Circle that touches the Curve in P or p.

72. The Curve Line describ'd by the Motion of any Point of the Circumf. of a Circle revolving on a Right, or Circular Base is called a *Cycloid*. Thus (when the Base is a Right Line) the Arcs EP, BQV, are those of the generating Circle, in their due places, whilst the Points P, V of the *Cycloid* are describ'd: So that (drawing PQ || CB) CE = arc EP = arc BQ, th. arc QV (BE) = QP.

Whence, to describe this Curve, make d : ½c :: BV : BC = Arc BQV, divide BC, and the arc BQV into the like Number of = parts, in the points E, e, &c. and Q, q, &c. draw BQ, Bq, compleat the Pgrs. BQPE, Bqπe, the Points P, π, will be in the Curve.

1. And where a part of some Curve VQ = x, and PQ



= y, then the Tangent Qt = $y\dot{x} \div \dot{y}$; But if x = Arc of a Circle, and $\dot{x} : \dot{y} :: m : n$, th. $\dot{x} = \dot{y}m \div n$, th. Qt = $ym \div n = x$. But in the *Primary*, *Protracted*, or *Contracted Cycloid*, $n =, >, < m$, Th. in the *Primary Cycloid* Qt = $y = x$. Th. ∠ tPQ (½∠tQA) = ∠VQA, *Th. the Tangent* Pt *to any Point* P *of the Cycloid is* || *to the respective Chord* VQ.

2. Th. QA : AV :: PA : AT; And PT : TA :: QV ($\sqrt{d}\times\sqrt{v}$) : VA ($\sqrt{v}\times\sqrt{v}$) :: $\sqrt{d} : \sqrt{v}$.

3. And (bec. VQ $= \sqrt{dv}$, and || PT) $v : \sqrt{dv} :: \dot{v} : d^{\frac{1}{2}} v^{-\frac{1}{2}} \dot{v} = \dot{z}$, th. the *Cycloidal Arc* VP $= (2\sqrt{dv})$ 2 VQ, i: e. *The Curve is = twice the corresponding Chord.*

4. *If the Semicycloid* CLd *be Evolv'd, the Curve describ'd will be a Semicycloid, Similar and Equal to the Evoluta.* For the Ray of the Curvature PL is a Tangent to the *Cycloid* in L, th. $\frac{1}{2}$ LP = $\frac{1}{2}$ *arc* LC = Cy = LE, th. EP = Cy (= BQ) but EP || Cy, th. (VB = and || CD) a *Circle* thro' B, Q, V, is = *Circle* DyC, and the *arc* BQ = *arc* Cy = yL = CE, Th. the Curve CP is a *Cycloid*, equal and like to CLd.

Th. if CDdk M be two like *Cycloidal Plates*, a Weight suspended at d, by the Chord dLP, when put in Motion, will describe a *Cycloid*, whose Axe is $\frac{1}{2}$ the Length of the describing *Pendulum*.

These things will also appear, by finding the Value of the *Ray of the Evoluta*, according to the general Rule (Art: 71.)

$$\text{Since } y = a + s, \text{ th. } \dot{y} = \overline{r\dot{v}} \div s + \overline{r\dot{v} - v\dot{v}} \div s = \overline{2r\dot{v} - v\dot{v}} \div s = \dot{v} \times \overline{\overline{2r - v} \div v}\Big|^{\frac{1}{2}}, \text{ th. } \ddot{y} = -$$

$$r\dot{v}^2 \div vs, \text{ th. PL } \left(\text{or } \frac{\overline{\dot{v}^2 + \dot{y}^2}\Big|^{\frac{3}{2}}}{-\dot{v}\ddot{y}}\right) = \frac{\overline{8r^3\dot{v}^6 \div v^3}\Big|^{\frac{1}{2}}}{-\dot{v}\ddot{y}} =$$

$$\frac{\overline{8r^3\dot{v}^6 \div v^3}\Big|^{\frac{1}{2}} \times vs}{r\dot{v}^3} = \overline{16r^2 - 8rv}\Big|^{\frac{1}{2}} = 2 \times \overline{4r^2 - 2rv}\Big|^{\frac{1}{2}}$$

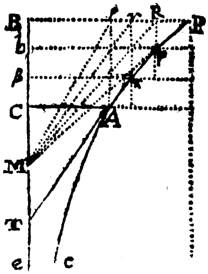
$$= 2 \times \overline{AQ^2 + AB^2}\Big|^{\frac{1}{2}} = 2QB = 2PE.$$

Th. if $v = 0$, PL = Vd $= (2 \times \overline{4r^2}|^{\frac{1}{2}})\ 4r = 2d$. And (since LP is Perp. to the Curve and || BQ) a Line PT || VQ shall be a *Tangent*. Also any part (CL) of the *Cycloidal Arc*, is = to twice (Cy) the corresponding Chord.

The *Cycloidal Space* (VCB) between the Curve and the Circle is = (bec. its Elements are = to their corresponding Arcs, which are an *Arith. Progr.*) $\frac{1}{2}$ ▭ CBV = $\frac{1}{2}$ *or* = *Generating Circle*.

73. Let two right Lines move at right ∠s, the one thro' equal Spaces in equal Times, the other thro' Spaces decreas-

decreasing in a *Geometric Progression* in the same Times, the Curve describ'd by their Intersection is called the *Logarithmic Line*. Th. its Ordinates to the Equal Divisions in the Axe are *Geometrically Proportional*. This Curve therefore may be describ'd thus; Divide BP continually, so that BP : BR :: *n* : *m*, by making Bb (= bβ = βC, &c.) $= \frac{1}{s}$ BM, and drawing bp, βπ, &c. || BP, also MP, MR, &c. then Rp, rπ, &c. drawn || BM will give the Points p, π, &c. in the Curve. By the Generation of this Line, 'tis plain.



1. If the *Ordinates* BP, bp, &c. are as Absolute Numbers, then Bb, Bβ, &c. increasing, or BM, bM, &c. decreasing, as the Absolute Numbers Decrease, are as their *Logarithms*, according to *Naper*'s, or *Briggs*'s Method.

2. Bec. MP = *Secant* of 45° to the *Radius* BM and *Tangent* BP, Th. the *Logarithmic Line* is composed of all the *Secants*.

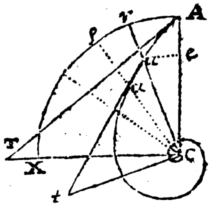
3. The Line BC continued, will be an Asymptote to the Curve PAc: And the Subtangent (s) or CT ($= y\dot{x} \div \dot{y}$) is every where the same; For Y : $\dot{Y}$ :: y : $\dot{y}$, and $\dot{x}$ (Cβ) is constant.

4. Since $\dot{x} = s\dot{y}$, th. Flux. of the Area ($\dot{x}y$) $= s\dot{y}$, Th. the Infinite Space ACec = sy = ▭ ACT = 2△ACT. Th. the Space between any two Ordinates Y, y, is $= s \times \overline{Y - y}$.

74. If whilst a Ray, with an Equable Motion, describes the Circumf. of a Circle, a Point from the Extremity thereof, moving towards the Centre with a Velocity decreasing in a Geometric Progression, will generate a Curve called the *Logarithmic Spiral*.

1. The Arcs Ar, rρ, &c. or the ∠s ACa, aCα, &c. being supposed infinitely small, and in an *Arith. Progression*; and the Radii, or Sides about those ∠s in a *Geometric Progression*, Th. the ∠s made by the Radii and Curve are every where equal.

2. If



2. If the Radii CA, Ca, &c. are as *Numbers*, the Arcs Ar, Aρ, &c. will be as their *Logarithms*.

3. If Ca $= y$, eA $= \dot{y}$, aA $= \dot{z}$, ta $= s$, then, because $m : n :: \dot{z} : \dot{y} :: s : y$, th. $m\dot{y} = n\dot{z}$, th. $my = nz$, $= ns$, th. $s = z$, i: e, *The Infinite Spiral* aC *is* = *to the Tangent* at.

75. The Ordinates in all Curves respect either the *Axis*, or a determined *Point*. And where the Ordinates are made parallel to one another, and right to the Axis; Let T = *Tangent*, τ = *Subtangent*, P = *Perpendicular*, ϖ = *Subperpendicular*, $z = x + \varpi$, and $a = \tau - x$. *Then*,

1. $(\dot{y} : \overline{\dot{y}^2 + \dot{x}^2}|^{\frac{1}{2}} :: y :)\ T = \overline{\dot{y}^2 + \dot{x}^2}|^{\frac{1}{2}} \times y \div \dot{y}$

2. $(\dot{y} : \dot{x} :: y :)\ \tau = y\dot{x} \div \dot{y}$

3. $(\dot{x} : \overline{\dot{y}^2 + \dot{x}^2}|^{\frac{1}{2}} :: y :)\ P = \overline{\dot{y}^2 + \dot{x}^2}|^{\frac{1}{2}} \times y \div \dot{x}$

4. $(\dot{x} : \dot{y} :: y :)\ \varpi = y\dot{y} \div \dot{x}$

5. $(x + \overline{y\dot{y}} \div \dot{x} =)\ z = \overline{y\dot{y} + x\dot{x}} \div \dot{x}$

6. (Bec. $x + a = y\dot{x} \div \dot{y}$) $a = \overline{y\dot{x} - x\dot{y}} \div \dot{y}$

The Values of any of these being made, from the given Conditions of a *Problem* proposed, respectively equal to their Values here, the Nature of the *Curve* will be found; and the contrary. Thus,

If $\tau = ny^n = y\dot{x} \div \dot{y}$, then $n\dot{y}y^{n-1} = \dot{x}$, th. $y^n = x$.

And if $y^n = x$, then $\tau\ (y\dot{x} \div \dot{y}) = ny^n = nx$.

Or if $px = y^2$, then $p\dot{x} = 2y\dot{y}$, th. $\varpi\ (= y\dot{y} \div \dot{x} = yp\dot{x} \div 2y\dot{x}) = \frac{1}{2}p$.

76. To find the *Length* (λ), or *Area* (α,) of any Curve Line, or Curvilinear Plane.

1. *Where*

1. *Where the Ordinates are parallel*; $\dot{\lambda} = \overline{\dot{x}^2 + \dot{y}^2}|^{\frac{1}{2}}$, and $\dot{\alpha} = \dot{x}y$.

2. *Where the Ordinates respect a Point*; (Let $r =$ Rad. of a Circle describ'd on that Point, $x =$ *Abscissa*, and $y =$ that part intercepted between the Point and Curve) $\dot{\lambda} = \overline{y^2 \dot{x}^2 + r^2 \dot{y}^2}|^{\frac{1}{2}} \div r$, and $\dot{\alpha} = y^2 \dot{x} \div 2r$. And by substituting in the room of $\dot{x}^2$ or $\dot{y}^2$, or of $\dot{x}$ or $\dot{y}$, their Values from the Equation of the Curve, there will be produced an Equation whose Fluent is the Length, or Area sought.

Thus in Case 1. If $y = x^n$, then $\dot{\alpha} = \dot{x}x^n$, th. $\alpha = (\frac{x^{n+1}}{n+1} = \frac{xx^n}{n+1} =) \frac{1}{n+1} xy$. And the *Asymptotic Space*, in the Hyperbola, (n being there *Negative*) is $= \frac{1}{-n+1} xy$; Therefore if $n <, =, > 1$, the *Space* will be *Finite*, *Infinite*, or *More than Infinite*.

More Examples are given where Occasion requires, th. are here, for Brevity's sake, omitted.

There are various other ways of finding the *Lengths*, or *Areas* of particular *Curve Lines*, or *Planes*, which may very much facilitate the Practice; as for Instance, in the *Circle*, the Diameter is to Circumference as 1 to $\overline{\frac{16}{5} - \frac{4}{239}} - \frac{1}{3}\overline{\frac{16}{5^3} - \frac{4}{239^3}} + \frac{1}{5}\overline{\frac{16}{5^5} - \frac{4}{239^5}} -$, &c. $=$ 3.14159, &c. $= \pi$. This *Series* (among others for the same purpose, and drawn from the same Principle) I receiv'd from the Excellent Analyst, and my much Esteem'd Friend Mr. *John Machin*; and by means thereof, *Van Ceulen*'s Number, or that in Art. 64. 38. may be Examin'd with all desireable Ease and Dispatch.

Whence in the *Circle*, any one of these three, a, c, d, being given, the other two are found, as, $d = c \div \pi = \overline{a \div \frac{1}{4}\pi}|^{\frac{1}{2}}$, $c = d \times \pi = \overline{a \times 4\pi}|^{\frac{1}{2}}$, $a = \frac{1}{4}\pi \times d^2 = c^2 \div 4\pi$.

And

And any *Segment of a Circle* = *Sector* — *Triangle* therein = $\frac{1}{2}ar - \frac{1}{2}sr$; But like Segments are as their respective Circles, i. e. π (the *Circle* where $r = 1$) : $\frac{1}{2}a - \frac{1}{2}s$ :: α or $\frac{1}{4}\pi^2 d$ (the *Circle* propofed) : $\overline{a - s} \times \alpha \div 2\pi$, or $\overline{a - s} \times d^2 \times \frac{1}{8}$; but a (of n degrees) in parts of s is $= \frac{1}{180}\pi n$, th. *Segment* $= \overline{\frac{1}{180}\pi n - s} \times \alpha \div 2\pi$, or $\overline{\frac{1}{180}\pi n - s} \times d^2 \times \frac{1}{8}$.

Or let v = *Versed Sine* or *Height of the Segment*, and s = *Sine* of n the double Co-Arc (in *degr.* and *parts*) to the *Sine* $\overline{1 - 2v \div d}$ (from the *Table of Nat. Sines*), then $\overline{\frac{1}{180}\pi n - s} \times \alpha \div 2\pi$ = *Segment.*

78. To find the Surfaces (s) of Solids generated by the Rotation of Planes about an Axis.

1. *Where the Axis of Rotation is the Abscissa,* $\dot{s} = \overline{\dot{y}^2 + \dot{x}^2}|^{\frac{1}{2}} \times \overline{c \div r} \times y$. Examples

In a *Cone*, $y = \frac{rx}{b}$, th. $\dot{s} = (\frac{c}{r} \times \frac{rx}{b} \times \sqrt{\frac{r^2 \dot{x}^2}{b^2} + \dot{x}^2}$ $= \frac{cx\dot{x}}{b} \times \sqrt{\frac{r^2 + b^2}{b^2}} = \frac{cx\dot{x}}{b} \times \frac{\sqrt{r^2 + b^2}}{b} =) \frac{cx\dot{x}}{b} \times \frac{l}{b}$, th. $s = (\frac{cx^2}{2b^2} l)\ \frac{1}{2}cl$ or $\frac{1}{2}\pi dl$ (when $x = b$).

Th. $\frac{1}{2}\pi dl : \frac{1}{2}\pi dr :: l : r$. And if the Side of the *Frustum* of a *right Cone* = b, its Curv'd Surface will be = $\overline{C + c} \times \frac{1}{2}b$. In a *Sphere*,

$$y = \overline{2rx - xx}|^{\frac{1}{2}}, \text{ th. } \dot{y}^2 = \frac{r^2 \dot{x}^2 - 2rx\dot{x}^2 + x^2\dot{x}^2}{2rx - xx}$$

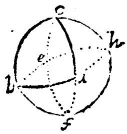
th. $\overline{\dot{y}^2 + \dot{x}^2}|^{\frac{1}{2}} = \sqrt{\frac{r^2 \dot{x}^2}{2rx - xx}}$ th. $\dot{s} = (\frac{c}{r} \times \overline{2rx - xx}|^{\frac{1}{2}}$ $\times r\dot{x} \times \overline{2rx - xx}|^{-\frac{1}{2}}) c\dot{x}$. Th. $s = cx$ (πdx the Surf. of the *Segment*) = (when $x = d$) cd or πd^2 the *Surf. of the Sphere.*

Th.

Th. $\pi dx : \pi d^2 :: x : d$. And $\pi dX - \pi dx : \pi d^2 :: X - x : d$. *Hence the Curved Surfaces of Segments, or Fruftums of Spheres, cut by parallel Planes, are equal to the correfponding Surfaces of the Sphere's Circumfcr. Cylinder.*

And $\frac{1}{n} cd : cd :: \frac{1}{n} c : c$, i: e. *The Surface of the Sphere contain'd between any two great Circles, is to the whole Surface, as the* ∠ *of Inclination, is to* 4 ∟*s.*

Th. $1 : r ::$ 3∠*s* (of any Sph. △ abc) — 2∟*s* (in parts of r): Surf. of the Sph. △. For △bac = feh, and △bec = fah; And 4∟: ∠cba + bca + bac :: ς (Surf. Sph.): bahcb + cbfac + abeca, i: e, 4∟ : 3∠s :: ς : 2△ + $\frac{1}{2}\varsigma$, or 2∟ : 3∠s :: ς : 4△ + ς, th. 2∟ ($\frac{1}{2}c$) : 3∠s — 2∟ (δ) :: ς : 4△, th. $2c$△ ($\delta\varsigma$) = $2rc\delta$, Th. △ = $r\delta$, th. &c.



2. *Where the Axis of Rotation is parallel to the Abfciffa,* (let the longeft Perp. from the Curve to the Axis of Rotation be p) $\dot{\varsigma} = \overline{p-y} \times \overline{\dot{y}^2 + \dot{x}^2}|^{\frac{1}{2}} \times \overline{c \div r}$.

3. *Where the Axis of Rotation is the Bafe of the Curve,* $\dot{\varsigma} = \overline{p-x} \times \overline{\dot{y}^2 + \dot{x}^2}|^{\frac{1}{2}} \times \overline{c \div r}$.

4. *Where the Axis of Rotation is a Tangent to the Vertex of the Curve,* $\dot{\varsigma} = \overline{\dot{y}^2 + \dot{x}^2}|^{\frac{1}{2}} \times x \times \overline{c \div r}$.

79. To find the Contents (χ) of Solids generated by the Rotation of Planes about an Axis.

1. *Where the Axis of Rotation is the Abfciffa, and the generating Plane adjacent to it, or to the Tangent at the Vertex.* (1) $\dot{\chi} = \dot{x}y^2 \times \overline{c \div d}$, or (2) $\dot{\chi} = \dot{x}\overline{p^2 - y^2} \times \overline{c \div d}$. *Examples* in *Cafe* 1. of this.

In a *Cone*, $y = \frac{rx}{b}$, th. $y^2 = \frac{r^2x^2}{b^2}$, th. $\dot{\chi} = \frac{cr^2x^2\dot{x}}{db^2}$, th. $\chi = \left(\frac{cr^2x^3}{3db^2} = \frac{1}{6}crb = \right) \frac{1}{12}\pi d^2 b$; Theref. *Cylinder* ($\frac{1}{4}\pi d^2 b$) : *Cone* ($\frac{1}{12}\pi d^2 b$) :: 3 : 1.

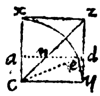
Or becaufe the Elements of a *Pyramid*, or *Cone*, are as the fq. Numbers from 0, th. their Sum, i: e. the *Pyramid*, or *Cone*, is $\frac{1}{3}$ of fo many times the greateft, i: e. $\frac{1}{3}$

 of

of the Circumſcr. *Priſm*, or *Cylinder*. Or ſup. B the Baſe of a ∟△, b, b, &c. its Parallels, $B \pm b = z$, or δ, Th. $b = B - \delta$, and $b^2 = B^2 - 2B\delta + \delta^2$, th. all the b^2s $= bB^2 - bB\delta + \frac{1}{3}b\delta^2 = b \times \overline{B^2 - B\delta + \frac{1}{3}\delta^2}$ $= b \times \overline{Bb + \frac{1}{3}\delta^2} = \frac{1}{3}\ b \times \overline{3Bb + \delta^2\ (\overline{B-b}|^2)} = \frac{1}{3}b \times$ $\overline{B^2 + Bb + b^2} = \frac{1}{3}b \times \overline{z^2 - Bb}$ = *Fruſtum of a ſq. Pyramid.* Th. the Fruſt. of any *Pyramid* or *Cone*, (A, a being the Areas of the Baſes) is $b \times \overline{A + \sqrt{Aa} + a} \times \frac{1}{3}$. If the Baſes are Circles, D, d, their Diameters, then the *Fruſt.* $=$ $b \times \overline{z^2 - Dd} \times \frac{1}{12}\pi$, and the whole *Cone* $= \frac{1}{12}\pi D^2 b$, as before.

If the *Fruſt.* of a *Cone* be cut by a Diagonal Plane paſſing thro' the op. Extremities of the Baſes, then the *Greateſt*, or *Leaſt Fruſt.* $= b \times \overline{\frac{1}{2}Dd + D^2 \text{ or } d^2} \times \frac{1}{12}\pi$; This is evident, as alſo, the preceding *Theorems*, upon Sight of a Fruſtum of a *Right Pyramid* of Four (or any even Number of Sides) reduced into its Component *Pyramids* and *Priſms*.

In the *Sphere*, $y^2 = dx - x^2$, th. $\dot{\chi}\ (dc\dot{x}x - cx^2\dot{x} \div d)$ $= c\dot{x}x - cx^2\dot{x} \div d$, th. $\chi = \frac{1}{2}cx^2 - cx^3 \div 3d = \frac{1}{2}\pi dx^2$ $- \frac{1}{3}\pi x^3 =$ (when $x = d$) $\frac{1}{6}cd^2$ or $\frac{1}{6}\pi d^3 = \frac{1}{6}d \times$ *Surf.* (πd^2,) Th. *Sph.* ($\frac{1}{6}\pi d^3$) : *Cylind.* ($\frac{1}{4}\pi d^3$) :: 2 : 3. Or ſince the *Ring* DE (= ⨀CE − ⨀AE = ⨀CA) = ⨀AN, all the *Rings* DEs = all the ⨀ANs, th. *Figure* YEXZY = *Cone* CXZ = $\frac{1}{3}$ *Cylind.* And *Hemiſph.* (*Cyl.* − *Cone*) = $\frac{2}{3}$ *Cylinder*. Th. *Cone*, *Sph.* and *Cylind.* are as 1, 2, and 3.



Or ſince $s^2 = c^2\ (dv) - v^2$, th. all the $s^2s = \frac{1}{2}v \times c^2$ $- v \times \frac{1}{3}v^2\ (= \frac{1}{2}\dot{v} \times \overline{s^2 + v^2}\ (c^2) - \frac{1}{2}v \times \frac{2}{3}v^2) =$ $\frac{1}{2}v \times \overline{s^2 + \frac{1}{3}v^2}$ or $\frac{1}{2}v \times \overline{⨀s + \frac{1}{3}⨀v}$, or $v \times \overline{\overline{2s}|^2 + \frac{4}{3}v^2} \times \frac{1}{8}\pi$, or $v \times \overline{3 \times \overline{2s}|^2 + 4v^2} \times \frac{1}{24}\pi$ = *Segment of a Sphere.* Th. $\frac{1}{2}$ *Sph.* $= \frac{1}{6}\pi d^2 r$, and *Sph.* $= \frac{1}{6}\pi d^3$, as before.

And

And if a *Sphere* be cut by any two Parallel Planes, S, s, shall be the Radii of the *Frustum's* Bases, its Height (V—v) call b, then $V = b + v$: but bec. any Segment of a Sph. $= \pi r v^2 - \frac{1}{3}\pi v^3$, Th. Fruft. $= \overline{2rv - v^2} \times \pi b + \overline{r - v} \times \pi b^2 - \frac{1}{3}\pi b^3 =$ (bec. $2rv - v^2 = s^2$, and $r - v = \overline{S^2 - s^2 + b^2} \div 2b$, by 65.23) $b \times \overline{S^2 + s^2 + \frac{1}{3}b^2} \times \frac{1}{2}\pi$.
Th. $v \times \overline{s^2 + \frac{1}{3}v^2} \times \frac{1}{2}\pi =$ *Segment*, and $\frac{1}{6}\pi d^2 r = \frac{1}{2}$ *Sphere*, as before.

In an *Oblong Spheroid*, let $a = \frac{1}{2}t = \frac{1}{2}$ the *Transverse*, $r = \frac{1}{2}$ the *Conjugate Axe*, then (bec. $a^2 : r^2 :: 2ax - x^2 : y^2 = \frac{2axr^2 - x^2r^2}{aa}$) $\dot{z} = \frac{2axr^2\dot{x}c - x^2\dot{x}r^2c}{a^2d}$, th. $z = \frac{r^2ax^2c - \frac{1}{3}r^2x^3c}{a^2d} =$ *Segment*, th. $\frac{2r^2ac}{3d} = \frac{1}{6}d^2\pi a = \frac{1}{2}$ *Spheroid*, and $\frac{1}{6}d^2\pi t =$ *Spheroid*. Th. *Cylind.* $(\frac{1}{4}\pi d^2 t)$: *Spheroid* $(\frac{1}{6}\pi d^2 t) :: 3 : 2$. And *Cone* ($\frac{1}{3}$ *Cylinder*) $= \frac{1}{2}$ *Spheroid*.

Or (sup. *Parameter* $= p$, and $p \div t = z$) since y^2 $(z \times \overline{2ax - x^2}) = z \times \overline{a^2 - \overline{a - x}|^2}$ or $za^2 - zm^2$, th. all the y^2s $= bz \times \overline{a^2 - \frac{1}{3}m^2} = \frac{1}{3}bz \times \overline{3a^2 - m^2} = \frac{1}{3}b \times \overline{2za^2 + za^2 - zm^2} =$ (bec. $y^2 = za^2 - zm^2$, and $r^2 = za^2$) $\frac{1}{3}b \times \overline{2r^2 + y^2}$, Th. $\frac{1}{3}b \times \overline{2\odot r + \odot y}$ or (if D, d be the Diameters of the Bases) $b \times \overline{2D^2 + d^2} \times \frac{1}{12}\pi =$ *Fruft. of a Spheroid* cut by two parallel Planes, the one passing thro' the Centre. Th. the *Spheroid* $= D^2 t \times \frac{1}{6}\pi$, as before.

If $y = x^n$, then $\dot{z} = \frac{c}{d}\dot{x}x^{2n}$, and $z = \frac{cx^{2n+1}}{\overline{2n+1} \times d}$

Th. if $m = \frac{1}{2}$, z $(= cx^2 \div 2d = cy^2x \div 4r) = \frac{1}{4}crb = d^2b \times \frac{1}{8}\pi =$ *Obtuse Parabolic Conoid*, Th. *Conoid* $(\frac{1}{8}\pi d^2 b)$: *Cylind.* $(\frac{1}{4}\pi d^2 b) :: 1 : 2$. And *Cone*, *Conoid*, and *Cylinder* are as 2, 3, and 6.

2. *Where the Axis of Rotation is the Base* (b) *of the Curve, and the Plane adjacent to it, or to the Tangent at the Vertex* (1) $\dot{\chi} = \dot{x}\,\overline{py - xy} \times c \div r$, or (2.) $\dot{\chi} = \dot{x} \times \overline{p - x} \times \overline{b - y} \times c \div r$.

Example in Case 1. If $y = x^n$, $\dot{\chi} = \overline{px^n\dot{x} - \dot{x}x^{n+1}} \times \frac{c}{r}$, th. $\chi = \frac{cpx^{n+1}}{r \times \overline{n+1}} - \frac{cx^{n+2}}{r \times \overline{n+2}}$. Th. if $n = \frac{1}{2}$, as in the common *Acute Parabolic Conoid*, then $\chi (= \frac{cx^2y}{\frac{3}{2}r} - \frac{cx^2y}{\frac{5}{2}r} = \frac{1}{3}\pi d^2y - \frac{1}{5}\pi d^2y) = \frac{2}{15}\pi d^2b$. Th. is to the *Cylinder* as 8 to 15.

Or in this Solid, call the Axe of the Curve B, its Parallels b, b, &c. and $B - b = \delta$ Th. $b = B - \delta$, and $b^2 = B^2 - 2B\delta + \delta^2$, th. all the b^2s $= bB^2 - \frac{1}{3}b \times 2B\delta + \frac{1}{5}b\delta^2 = b \times \overline{B^2 - \frac{2}{3}B\delta + \frac{1}{5}\delta^2} = \frac{1}{3}b \times \overline{2B^2 + \frac{3}{5}\delta^2 + (B^2 - 2B\delta)b^2 - \delta^2} = \frac{1}{3}b \times \overline{2B^2 + b^2 - \frac{2}{5}\delta^2}$ or $\frac{1}{3}b \times \overline{2\odot B + \odot b - \frac{2}{5}\odot\delta}$ = Frust. of an *Acute Parabolic Conoid*, = (if D, d be the Diameters of the Bases) $\overline{2D^2 + d^2 - \frac{2}{5}\delta^2} \times b \times \frac{1}{12}\pi$.

3. *Where the Axis of Rotation is a Tangent at the Vertex of the Curve, and the Plane adjacent to it, or to the Abscissa.* (1.) $\dot{\chi} = \dot{y}x^2 \times c \div d$, or (2) $\dot{\chi} = xy\dot{x} \times \overline{c \div r}$. So that (in Case 2.) if $y = x^n$, $\dot{\chi} = \frac{c}{r} \times \dot{x}x^{n+1}$, and $\chi = \frac{cx^{n+2}}{r \times \overline{n+2}}$, Th. if $n = +\frac{1}{2}$ (as in the *Common Parabola*) $\chi = (\frac{2cx^{\frac{1}{2}}x^2}{5r} =) \frac{1}{5}\pi d^2b$; Th. is to the *Cylinder* as 4 to 5. And the Solid generated by the Infinite *Asymptotic Space* (in the *Apollonian Hyperbola*, where $y = x^{-1}$) about that *Asymptote*, is $= \frac{1}{2}\pi dxy$; Th. is to the *Cylinder* (the *Rad.* of whose Base is x, and *Alt.* $= y$) as 2 to 1. *So that, in this Case, an Infinite Space generates a Solid of a Finite Dimension.*

4. *Where*

4. *Where the Axis of Rotation is parallel to the Abscissa, and the Plane adjacent to it, or to the Abscissa*; (1) $\dot{\chi} = \overline{\dot{x}p - \dot{y}}\big|^2 \times \overline{c \div d}$, or (2) $\dot{\chi} = \overline{\dot{x}p^2 - p - \dot{y}}\big|^2 \times \overline{c \div d}$.

Note, Where the Ordinates are Oblique to the Diameter, or Axis of Rotation, or where they respect a determined Point; they are to be reduced to other equivalent ones, parallel between themselves, and right to the Axis of Rotation.

Hence in *Cask Gauging*, let *B*, *H*, be the *Bung*, and *Head Diameters*, *L*, the *Length*, *n* the Number of *Cubic Inches* in the *Gallon*; and the *Diam.* : *Periph.* : : 1 : π.

Then the Contents of a Cask taken as the middle Frustum of an *Acute Parabolic Conoid*, or of an *Oblong Spheroid*, is

$$\overline{L \times \overline{2B^2 + H^2 - \tfrac{2}{5}\overline{B - H}\big|^2} \div \frac{12n}{\pi}}, \text{or } \overline{L \times \overline{2B^2 + H^2} \div \frac{12n}{\pi}}$$

Th. in a Spheroidal Cask, when not Full, the Axis being Perpendicular to the Horizon, $w =$ wet part of *L*, $d = \frac{1}{2} L \backsim w$; if w be $>$, or $< \frac{1}{2} L$, the Cask is more, or less than $\frac{1}{2}$ Full by $\overline{\overline{3B^2 d - d^3} \times \overline{B^2 - H^2} \div \tfrac{1}{4}L^2} \div \frac{12n}{\pi}$, and therefore the Liquor contain'd therein must be

$$\overline{\tfrac{1}{2} L \times \overline{2B^2 + H^2} \pm \overline{3B^2 d \mp d^3} \times \overline{B^2 - H^2} \div \tfrac{1}{4}L^2} \div \frac{12n}{\pi}$$

But in a Cask not Full, whose Axe lies parallel to the Horizon, and the Liquor cutting the Head; Let $w =$ wet part of *B*, S $=$ *Segment* of a *Circle* (whose *Area* $= 1$) to the *versed Sine* $w \div B$, and C $=$ Content of the Cask, then will SC be the Quantity of Liquor remaining.

80. *Of* Projection.

The Impression or Representation of a Surface, on a Plane is called the *Projection* of that Surface: If the Impression be made by parallel Lines, 'tis called an *Orthographic Projection*; But if by Lines intersecting in the same Point, 'tis called a *Stereographic Projection*.

1. In

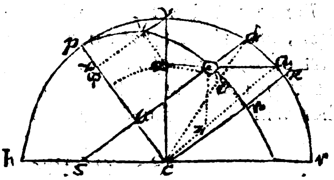
1. In the *Orthographic Projection*, 'tis evident that, 1. A right Line Perpendicular, Parallel, or Oblique to the Orthographic Plane is projected into a Point, an Equal, or a Shorter Line: And parallel Lines, tho' oblique to the Plane, are parallel when projected. 2. A Surface perpendicular to the Plane is projected into a right Line, equal to that intercepted between the projecting Lines: But a Surface parallel to the Plane, into an equal and like Surface. 3. Any *Conic Section* Oblique to the Plane is projected into another of the same Kind; Hence a *Circle* into an *Ellipse*. 4. The Representation of any Point of the Surface of the Sphere, *Orthographically* projected upon a Plane bisecting it, is distant from the Centre, by the *Sine* of the Arc from the Point to be projected to the Vertex of that Sphere.

Hence $c\alpha$ is Equivalent to $r\alpha$, $c6$ to $æd$. And all *Circles Perpendicular*, or *Parallel* to the Plane, are projected into *Right Lines*, or *Equal Circles*; but all other into *Ellipses*. Theref. all right Lines passing thro' the Centre of this Projection, are measured from the *Sines*; and all others are reduced to such; As for Instance,

Given the Latitude of the Place, Sun's Declination and Altitude; Reqd. the Sun's Azimuth, and Hour of the Day; the Sphere being *Orthographically* projected upon the Plane of the *Meridian*.

Let $ræ =$ *Co-Latitude*, ra *or* $c\alpha = \odot$'s *Altitude*, $æd$ *or* $c6 = \odot$ *Declination*: *Draw* $a\alpha$, $d6$ *the Parallels of Alt. and Decl. draw* $\odot z$, $\odot u \parallel \alpha c, 6c$; *then* cz, $cu =$ *Sines of the like* Azimuth *from the West, or* Hour *from* 6, $\odot\alpha$, $\odot 6$.

To describe the Representation of *Oblique Circles* on this *Projection*; As suppose, to describe a *Meridian*, passing thro' the *n*th. Degree of the *Equator*, on the *Analemma*.



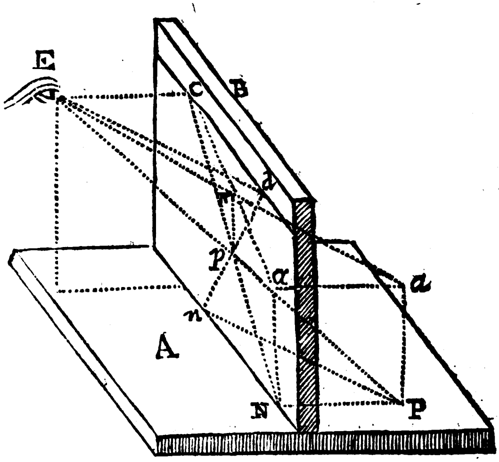
'Tis but making $cæ : cn :: \chi v : \chi \nu :: 6d : 6\odot$, and the Points ν, $\odot$, are in the *Ellipse* sought. Or the whole *Ellipse* (according to the Method in *Art.* 68.) is readily drawn by means of the *Foci*, (φ, t) found by cutting the Transverse Axe from n with pc its half.

2. In

2. In the *Stereographic Projection*, that Point in which all *Lines* from the Extremes of the thing projected do concur is called the *Projecting Point*; those Lines the *Projecting Lines*; the Plane cutting them, the *Stereographic Plane*, or *Plane of Projection*: And the Stereogr. Plane is supposed (if not otherwise mention'd) perpend. to another which is also supposed paral. to the Horizon, having thereon, the *Seat* or *Ichnography* of the thing to be projected, and called the *Original* or (by some) the *Geometrical Plane*. A Plane passing thro' the projecting Point, || to the Horiz. or perp. to it and to the Pl. of Proj. is call'd the *Horizontal*, or *Vertical Plane*. The com. Sect. of the Horiz. Plane, and Plane of Proj. is called the *Horizontal Line*, And the com. Sect. of the Vertical and Horiz. Pl. is called the *Axis of Projection*. The Intersect. of the Axis of Project. and the Horizontal Line is called the *Centre of Projection*. The com. Sect. of the Stereographic and Original Plane is called the *Base-Line*. The Intersect. of the Horiz. Line, and another drawn from the projecting Point parallel to a given right Line (L) is called the *Projecting Centre* of the Line L.

In this *Projection*, 'tis manifest, that, 1. The Projection of a Point, or Right Line is a Point, or Right Line. 2. The Projection of a right Line, on or above the Original Plane, paral. to the Base-Line is parallel to it in the Pl. of Projection: And the parts of its Projection, are :: to those of the Line it self. 3. The Projection of a right Line ⊥ or inclin'd to the Original Pl. and || to the Pl. of Proj. is ⊥ or so much inclin'd to the Base Line: And the Projections of the equal parts of a Line perp. to the Pl. of Project. are unequal; and those of the most remote are the smallest. 4. The Projections of parallel right Lines on or parallel to the Orig. Pl. and inclin'd to the Plane of Project. being produced, shall all pass thro' the same Point in the Horiz. Line. 5. Theref. the Projection of right Lines perpend. to the Pl. of Project. do pass thro' the Centre of Projection. 6. The Projections of equal Lines perp. or equally inclin'd the same way, on the same right Line (L) inclin'd to the Pl. of Projection, are limited by two Lines, which being prolong'd, shall pass thro' the Projecting Centre of the Line L. 7. The Projections of Points above, or below the Horiz. Plane, are above, or below the Horizontal Line; and the more remote

remote they are, the lower, or higher they be. 8. Given, *the Height of the Projecting Point above the Original Plane, with its Distance from the Plane of Projection, as also the Dist. of the Point to be projected from the same; Reqd. the Projection of that Point.* 1. If the Point to be projected be on the Original Plane; Let E, be the *Proj. Centre*,& B,C, the *Plane*,and *Centre of Projection*; P a Point in the Orig. Plane A, whose Projection p is sought. Set EC on the Horizont. Line, and PN on the Base-Line, the contrary way, as from C to d, and from N to n, draw nd, CN, their Intersect. p is the Projection of P reqd.

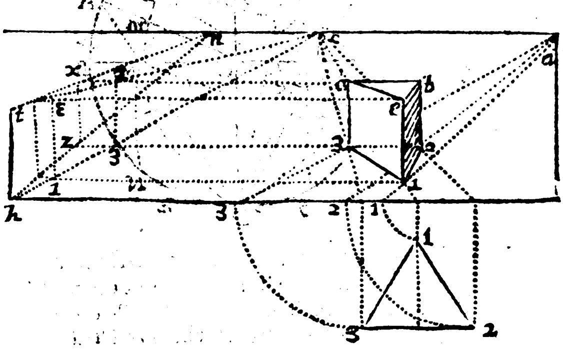


For (bec. Ed || Pn) d is the Projecting Centre of the Line Pn, but the Project. of P is in nd (by 4) and in NC (by 5) th. must be in their Intersect. p. 2. If the Point to be projected be (*a*) above the Original Plane (A) Find P its *Seat*, and N the Point of Incidence of P, erect Nα = P*a*, then will CN, Cα be the produced Projections of PN, *a*α (by 5,) and p the Project. of P (by the last,) th. (drawing pπ || Nα) π will be the Project. of *a* sought,

These are the Principles on which the Practical Part of Common *Perspective* depends; and by which 'tis easie to give the true Representation of any Object, whose Situ-

Situation, with Respect to the Eye and to the Plane on which 'tis to be represented are, given.

As in the following Instance, where 123 is the Base of a *Triangular Prism*, whose Altitude is *bt*, to be put into Perspective; *c* the *Centre of Projection* or *Point of Sight*; *d* the *Projecting Centre* or *Point of Distance*: The Pra-

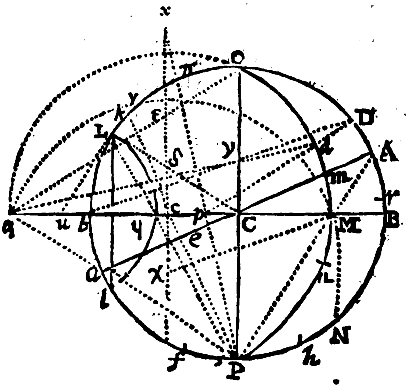


ctice is evident from the foregoing Rules.

Note, That 'tis indifferent what Point of the Horizontal Line be taken, for determining the Elevations; it may as well be at *n* as at *c*; for zx || α3, bec. nx : xt : : (cα : αt : : c3 : 3h : :) nz : zh, th. zx (|| ht) || α3.

9. The *Stereographic Projection* of any Point (A) of the Surface of a *Sphere*, from the Pole of a Plane bisecting it, upon that Plane, is distant from the Centre by the $\frac{1}{2}$ Tangent of the Arc from the Point to be projected to the Pole opposite to the projecting Point; Thus, M is the Projection of A; th. BM *equiv.* BA, MC *equiv.* Ao, and cδ *equiv.* Md.

1. Here all Circles, passing thro' the projecting Point, are projected into *Right Lines*; But all others into Circles: Since the Plane of the Projection cuts the Surface of the Cone form'd by the projecting Lines, either parallel, or (bec. ∠BqP = $\frac{1}{2}$*arc* BP — $\frac{1}{2}$*arc* ba (by 54) = $\frac{1}{2}$*arc* Pa = ∠aAP) subcontrary to its Base. And the Angles in this Projection are equal to their correspondent ones on the Sphere.



2. If Aa be the Diameter of a *great Circle* to be projected, whose Obliquity is AB (or ω); Let oπ = AB = πk, draw P$p\pi$, Pck, p shall be the *Upper Pole* (bec. πA = Bo = 90°,) and c the *Centre* of the *Projected Oblique Circle* PMo; since cM (= t, ω + t, $\overline{\frac{1}{2}90-\frac{1}{2}\omega}$ = s, ω (by 65.16) = cP. And if Pb = AB or oπ, a right Line thro' P and b will project the *Under Pole* in the produced Plane.

Also the Centre of a Lesser Circle (Lyl) perpendicular to the Plane, is found by drawing CL, and Lu ⊥ to it, then will u be the Centre reqd. Since uy (= s, ω — t, $\overline{\frac{1}{2}90°-\frac{1}{2}\omega}$ = t,ω (by 65.16) = Lu.

The *Radius* and *Centre* of any Circle whose Distance from its Pole is d, may easily be found thus; From (P) the projected Pole, on a great Circle (whose Centre is c) passing thro' it, d being set off, finds a Point (p), whose Tangent, terminated at π in a Line thro' c and P, or (which is the same) a Tangent to P, terminated at π in a Line thro' c and p, shall be the *Radius*, and πc the Distance from the *Centre* required.

3. The Centre χ of any great Circle (NMq) passing thro' M, shall be in cx, the produced Diameter of a Circle (PMo) bisecting the Primitive in P and o; And (bec. Sph. ∠ PMN = ∠ cMχ by 56) cχ = Tangent of the Sph. ∠ PMN to the *Rad.* Mc.

4. *Hence to draw the Circumf. of a great Circle thro' any two Points, (m, n,) one, or both within the Primitive.* Thro'

(m)

(m) one of the Points in the Circle, draw a Diameter Aa, and πb ⊥ to it, draw π m, let bf = 2Ar, a Ruler on π and f cuts Aa in e, thro' e draw sv ∥ πb, a right Line (Bb,) bisecting the Distance between m and n at ∟s, will cut sv in the Centre sought. For the Centres of all great Circles passing thro' m are in the Line sv, and the Centre of a Circle passing thro' m and n is in the Line Bb, th. the Centre reqd. must be in both, i. e. in their Intersection c.

5. The Circumf. of a great Circle in this Projection, is Divided, or Measured, by a Ruler laid on either Pole, and the Divisions of the op. ½ Periphery of the Primitive; For, since Bo *equiv.* Mo *equiv.* Co *equiv.* cε *equiv.* 90°. th. BD (*equiv.* Cγ *equiv.* cδ) *equiv.* Md. And the Circumf. of a small Circle (whose Dist. from its Pole is d) may be divided by Lines from that Pole to the Divisions in the Circumf. of a small Circle ∥ to the Pl. of Proj. and descr. at the Dist. d from the under Pole of that Plane.

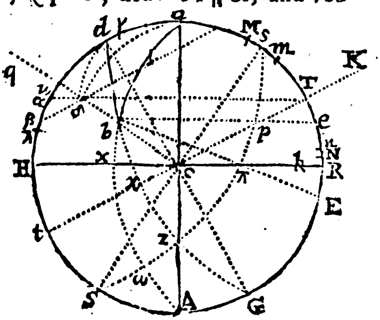
Or any Projected Circumf. is divided by Lines from the Divisions of its produced ½ Periphery to either Pole.

6. A Spheric ∠ or the Inclination of the Planes of two great Circles is equal to the Distance of their Poles, or (which is the same) is measured by an Arc of a great Circle whose Pole is the Angular Point; and theref. is so in the Projection. And the Circumfs. of great Circles intersecting at ∟s, do mutually pass thro' each other's Poles.

7. The △adb, and △adδ are the *Stereographic* and *Orthographic* Projections of the same Sph. △; and their ∠s b and δ are in the same Radius cu. Let scS perp. cu.

Given b, *Reqd.* δ; draw Sbv, and vδ ∥ sS.

Given δ, *Reqd.* b; draw δu ∥ Ss, and vbS.



8. Let $\nu\lambda = \nu s$, draw $S\lambda q$, with qs, from q, descr. a Circumf. $sp\pi S$ cutting cT, cR, db, ab, in p, π, z, ω, then will p, π, c, be the Pole of db, ab, ad; and b the Pole of $s\pi S$: Draw bpe, $b\pi E$, then $Ee = \pi p = Sph. \angle b$.

9. Draw $A\pi m$, GpM, make mn = ma, and MN = Md, a Line thro' A and n, G and N gives k, K, the Centre of the Arc abA, dbG.

10. Draw $\delta\alpha, \delta\beta \parallel$ HR, tT, then will $H\alpha, t\beta$ *equiv.* bx, $b\chi$.

11. And in the Sph. $\Delta c\pi p$, $\angle c =$ ad; $\angle p = \chi z =$ db *equiv.* $d\beta$; $\angle\pi =$ *Supl.* $x\omega$ = *Supl.* ab = Ab *equiv.* $A\alpha$, *i.e.*

$$\left\{ \begin{matrix} \angle c, \angle p, \angle \pi \\ p\pi, c\pi, cp \end{matrix} \right\} \text{ in } \Delta c\pi p, \text{ is} = \left\{ \begin{matrix} ad, db, Supl.\, ab \\ \angle b, \angle a, Supl. \angle d \end{matrix} \right\} \text{ in } \Delta abd.$$

12. Hence (bec. ad + $d\beta$ + $A\alpha$ = 2 ∟s + $\alpha\beta$) the 3 $\angle$s of a Sph. Δ is greater than 2 ∟s.

10. The Projection of any Point (P) of a Hemisphere, from the Centre (C) of the Sphere, upon a Plane touching it (in T) is distant from the Point of Contact (T) by the Tangent of the Arc PT; And this is called the *Gnomonic Projection*; where 'tis plain, that, 1. All great Circles on the Sphere, perpendicular to the Plane, are projected into *Right Lines*; all others, (as they are *parallel* to, *touch*, *cut*, or *neither*, a great Circle parallel to the Plane) into *Circles*, *Parabola's*, *Hyperbola's*, or *Ellipses*.

Hence, if the Sun moves in a Circle parallel to the Equator, the Shade of the Centre will describe a Conic Surface; And therefore, upon a Plane parallel to a great Circle, *touching*, or *cutting* the Conic Surface, or *neither*, will describe the Curve of a *Parabola*, *Hyperbola*, or *Ellipse*. Theref. on any Plane where the *Sun sets*, *does not set*, or *only touches*, the Projection of any Parallel of Declinaon i an *Hyperbola*, *Ellipse*, or *Parabola*.

Thus, if *ce*, *ca* be the *Stile*, and *Substile*, *δec* or *dec* the *Co-declinat.* of the *Sun*; then,

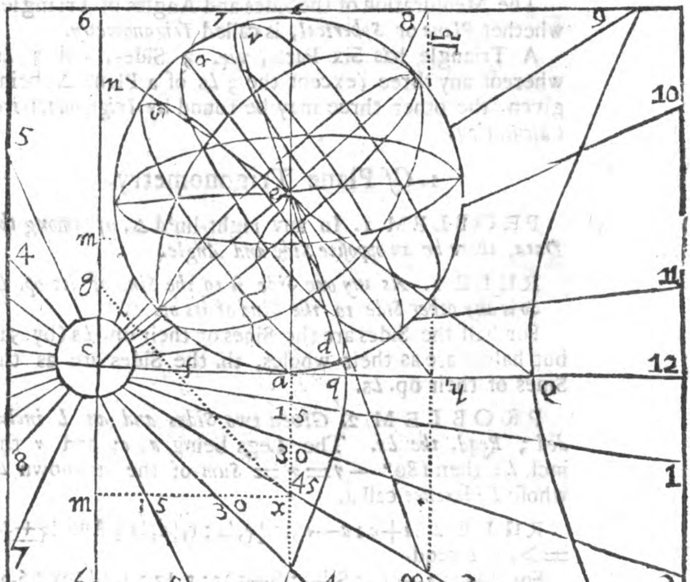
If *eδ* produced be parallel to the Plane, and *ed* produced cut it in *q*, let *en* = *eq*, from *n* with *nq* cut *qc*, the Projected *Meridian* produced, in Sup. π, then πq = *Parameter*, and $\frac{1}{4}\pi q$, set from *q* the *Vertex*, gives the *Focus f* of the *Parabola*: If *ed* produced cut *ca* in *q*, and *eδ* produced cut it in *Q*, *above*, or *below* the Vertex *e*, let *en* = *eq*, bisect *Qq* in *y*, make $yf = y\varphi = \frac{1}{2}nQ$, then shall *f*, φ be the *Foci* of the *Ellipses*, or opposite *Hyper-*

bola's

bola's described by the Shade of the Point *e*, their *Transverse Axe* being *Qq*, and *Distance of the Foci nQ* (by 69. 6.)

Hence the Parallels of the Sun's Declination, &c. in the *Gnomonic Projection*, are easily described (by 68.) upon an Horizontal Plane, in any Latitude.

And all Planes are parallel to some great Circles, which also are *Horizons* in some place or other; therefore all Planes on which Dyals are made, are *Horizontal Dyals* on some part of the Earth.



The Method of projecting the *Hour Circles* Gnomonically, on an Horizontal Plane, &c. or the Practice of *Dyalling* is hence also evident; Thus, let *c*12 be the Projection of the *Meridian*, and (*c*6 ⊥ to it,) that of the 6th *Hour Circle*; make *Lace* = *Stile's Height*, and *af* = Sine thereof, to any Rad. *ca*; let *ax* = *cm* = *af*, on *ax*, *mx*, from *a*, *m*, set the Tangents of the Hour ∠ at the Pole, from 12 to 3, and 6 to 3, to the Rad. *ax*, *mx*; Lines drawn

drawn from *c* thro' those Points, shall be the *Hour-Lines* reqd.

For $\begin{cases} ca : r :: ax : t, acx. \textit{ and } cm : r :: mx : t, mcx \\ fa : r :: ax : t, afx. \textit{ and } gm : r :: mx : t, mgx \end{cases}$
th. ca : fa :: t, afx : t, acx. *and* cm : gm :: t, mgx : t, mcx i : e. *Rad.* : S, *Stile's Height* :: t, *Hour* ∠ *at the Pole* : t, *Hour* ∠ *on the Plane.*

81. *Of* Trigonometry.

The Mensuration of the Sides and Angles of Triangles, whether *Plane* or *Spherical*, is called *Trigonometry.*

A Triangle has Six Parts, *viz.* 3 Sides, and 3 ∠s, whereof any three (except the 3 ∠s of a Plane △) being given, the other three may be found by *Trigonometrical Calculation.*

1. *Of* Plane Trigonometry.

PROBLEM 1. In any right-lin'd △, *if among the Data, there be an opposite Side and Angle.*

RULE 1. *As any one Side is to the Sine of its op.* ∠. *So is any other Side to the Sine of its op.* ∠.

For half the Sides are the Sines of their op. ∠s (by 51) but halves are as their wholes, th. the Sides are as the Sines of their op. ∠s.

PROBLEM 2. *Given two Sides and an* ∠ *included; Reqd. the* ∠*s.* The Legs being *a*, *c*; and *y* the incl. ∠; then $180° - y = z =$ *Sum* of the unknown ∠s whose *Difference* call *d*.

RULE 2. $a + c : a - c :: t, \frac{1}{2}z, \angle s : t, \frac{1}{2}d, \angle s$; And $\frac{1}{2}z \pm \frac{1}{2}d = >, <$ ∠ reqd.

For $a + c : a - c$ (:: S+s : S−s) :: t, $\frac{1}{2}$z : t, $\frac{1}{2}d$. (by 66.2)

Note, If ∠*y* = ∟, then *a* : *r* :: *c* : t, op. ∠, or *c* : *r* :: *a* : t, op. ∠.

For the *Tabular Parts* of any two Lines are proportional to their Parts according to any other Measure.

PROBLEM 3. *Given the three Sides; Reqd. the* ∠*s.* Let *a*, *c* be the Legs of the ∠ reqd. and $a + c = z$, or *d*, call the Base or Side op. *b*, and the Diff. of the Segments of *b*, made by a perp. from the ∠ *sought*, call δ.

RULE

RULE 3. $b : z :: d : \delta$. (For $p^2 + n^2 + \delta^2 + 2n\delta = \overline{p^2 + n^2}\,(a^2) + d^2 + 2ad$, Th. $2n + \delta : 2a + d :: d : \delta$) And $\frac{1}{2}b \pm \frac{1}{2}\delta =$ $>, <$ *Segment*; whence the $\angle$s may be found by *Probl.* 1.

Or bec. $r : s\ (:: c : x) :: 2a \times c : (2a \times x) \pm a^2 \pm c^2 \mp b^2$ (by 64. 17) th. $2ac : (2ac \pm a^2 \pm c^2 \mp b^2)\ z^2 - b^2$, or $b^2 - d^2 :: r : (r \pm s)\ v$, or $v :: \frac{1}{2}r^2 : (\frac{1}{2}rv)\ s^2, \frac{1}{2}\angle$, or $(\frac{1}{2}rv)$ $s^2, \frac{1}{2}\angle$ (by 65. 14 & 12) th. $4ac : r^2 :: z^2 - b^2 : s^2, \frac{1}{2}\angle :: b^2 - d^2 : s^2, \frac{1}{2}\angle$. And $z^2 - b^2 : b^2 - d^2 :: s^2, \frac{1}{2}\angle : s^2, \frac{1}{2}\angle :: r^2 : t^2, \frac{1}{2}\angle$ (by 65. 10) $:: t^2, \frac{1}{2}\angle : r^2$ (by 65. 5) Let $b =$

$\frac{1}{2}\overline{a + b + c}\ (\frac{1}{2}\overline{z + b}.)$ then will these *Rules* follow, *viz.*

1. $(4ac : \overline{z + b} \times \overline{z - b} ::)\ ac : b \times \overline{b - b} :: r^2 : s^2, \frac{1}{2}\angle$.

2. $(4ac : \overline{b + d} \times \overline{b - d} ::)\ ac : \overline{b - a} \times \overline{b - c} :: r^2 : s^2, \frac{1}{2}\angle$

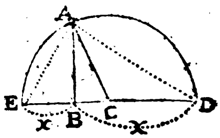
3. $(\overline{z + b} \times \overline{z - b} : \overline{b + d} \times \overline{b - d} ::)\ b \times \overline{b - b} : \overline{b - a} \times \overline{b - c} :: r^2 : t^2, \frac{1}{2}\angle$

4. $(\overline{b + d} \times \overline{b - d} : \overline{z + b} \times \overline{z - b} ::)\ \overline{b - a} \times \overline{b - c} : b \times \overline{b - b} :: r^2 : t^2, \frac{1}{2}\angle$

And if $L =$ *Logarithm Sine*, and $l =$ *Arith. Compl.* of any *Log. Sine*, then, for Practice,

$\overline{l,a + l,c + L,\tfrac{1}{2}\overline{b + d} + L,\tfrac{1}{2}\overline{b - d}} \div 2 = L, \frac{1}{2}\angle$ reqd.

To the various Uses of *Plane Trigonometry*, may be added, that,



If an *Equation* be $x^2 \pm ax = b^2$; Make b, and $\frac{1}{2}a$ the *Legs* (AB, BC) of a ∟△, Then *Hypotenuse* $\mp \frac{1}{2}a$ (i: e EB, or BD) $= x$. Or bec. $b : r :: \frac{1}{2}a : t, \frac{1}{2}\angle$, Th. $r : b :: t, \angle : x$, or $s, \angle : b :: s, \angle : x$.

If the *Equation* be $-x^2 - ax = b^2$; Make b, and $\frac{1}{2}a$ the *Perp.* and *Hypot.* (AB, AC) of a ∟△, then *Hypotenuse* $\pm$ *Base* (i: e. BD, BE) $=$ *Greatest*, or *Least Root.* Th. bec. $\frac{1}{2}a : r :: b : s, \frac{1}{2}\angle$, th. s, or $s, \angle : b :: s$, or $s, \angle : x$ the $<$ or $>$ *Root.*

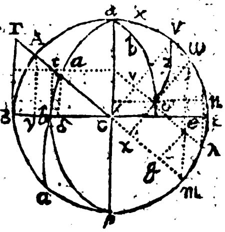
2. *Of* Spheric Trigonometry.

In ∟d Sph. △ CAB, Cab, *Orthographically* projected, sup. H, P, & B, the *Hypotenuse*, *Perpendicular*, and *Base*.

1. { s, *H* : s, *P* : : s, *b* : s, *p* : : R : s, } ∠ at the *Base*.
   { s, *B* : t, *P* : : s, *b* : t, *p* : : R : t, }

For cA : Aγ : : ca : aδ ; And cB : TB : : cb : tb.

2. In any Sph. △ (bαβ), *The Sines of the Sides are as the Sines of their op.* ∠*s*. For (by *prec.*) R : s, β : : s, βb : s, Bb, and R : s, α : : s, αb : s, Bb; Th. s, β : s, α : : s, αb : s, βb.



3. Theref. in Right ∠d Sph. △s, of the three Parts which (besides the ∟) enter the Question, let that be called the *Middle* to which both *Extremes* are either *Conjunct* or *Disjunct*; Then, *Rad.* + *Log. of the Middle* = *Logs. of the Extremes.*

If on	*Middle.*	*Extr. Conj.*	*Extr. Disj.*
Hyp. or ∠	*Co-Sine.*	*Co-Tangent.*	*Sine.*
Leg.	*Sine.*	*Tangent.*	*Co-Sine.*

Hence all the *Cases* are easily solved.

For the *Middle part* may be one of the *Legs* (*b*, *p*;) the *Compl.* of the *Hypotenuse* (*h*;) or the *Compl.* of either of the *Oblique* ∠s (*e*, *c*;) In each, there are three *Cases*: As sup. the *Middle*, be *b*, ∠*e*, *h*.

1. (*r* : t, ∠*e* : :) s, *b* : t, *p* (by 1) : : *t*, ∠*e* : *r* (by 65. 5)
2. (*r* : t, β (t, *b*) : : s, ε) *s*, *e* : (t, *n*) *t*, *b* : : t, *b* : *r* (by 1.)
3. (*r* : t, ∠*c* : : s, *n*) *s*, *h* : (t, ε) *t*, *e* (by 1) : : *t*, *c* : *r* (by 65. 5)

Th. R×S, *Middle* = ▭ *Tangents of the Conj. Extremes.*

1. *r* : (s, *b*) *s*, *n* : : (s, ∠*c*) *s*, χ : s, *b* (by 1)
2. *r* : (s, ∠*c*) *s*, χ : : (s, π) *s*, *p* : (s, ε) *s*, *e*
3. *r* : (s, π) *s*, *p* : : (s, β) *s*, *b* : s, *n* (*s*, *h*)

Th. R × S *Middle* = ▭ *Co-Sines of the Disj. Extremes.*

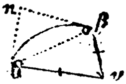
Note, In ∟ Sph. △, if the *Legs* (and th. the ∠*s*) be *like*, or *unlike*, the *Hypotenuse* is <, or > 90°. and the contrary.

If the *Hypotenuse* is < or > 90°, either *Side* will be *like*, or *unlike* its adjacent ∠; and the contrary.

In *Oblique* ∠*d Spherical Triangles*; Suppose the Obl. △ to be divided into ∟*d* △*s*, by a *perp.* let fall from one End of a given Side, (whose other End is adjacent to a given

given ∠) *within*, or *without* the △, as the ∠s at the Base are of the *same*, or of *different* Kind. Then, if there are two parts *given* in one △, corresponding to the two *given* and *sought* in the other, the *perp.* being a 3d. in each, will denote a *Middle* or *Extreme Part*; *And* (rejecting the *Perp.*) *the Logarithms of the op. parts are equal.* But if 4 such Parts don't correspond (as it happens when the *perp.* falls *from* or *on* a part *given* or *required*;) find (by 3) a part (in the ⌞d△ wherein a Side and an ∠ are given) of the *given* or *sought* one, *from* or *on* which the *perp.* falls, Then you'll have the 4 parts reqd. for the *Second Operation.*

When two Sides, or ∠s, and the ∠, or Side included, are given; the other ∠s, or Sides may be found, without letting fall a perp. by these Proportions.



1. $s,\frac{1}{2}z,cr : s,\frac{1}{2}d,cr :: ct,\frac{1}{2}\angle : t,\frac{1}{2}d,\angle s$.
2. $cs,\frac{1}{2}z,cr : cs,\frac{1}{2}d,cr :: ct,\frac{1}{2}\angle : t,\frac{1}{2}z,\angle s$.

For, let $vb\beta$ be the *Obl. Sph.* △ *Stereographically projected*, if the Right Line $b\beta$, and the Tangents bn, βn, be drawn, it appears that the *Sph.* ∠ $v\beta b - vb\beta =$ *Plane* ∠ $v\beta b - vb\beta$, Th. ($z,cr : d,cr$ ::) $cr,\frac{1}{2}\angle v : t,\frac{1}{2}d,\angle s$ (by 81, 1. Pr. 2.) :: ($z,\frac{1}{2}t,cr : d,\frac{1}{2}t,cr$ ::) $s,\frac{1}{2}z,cr : s,\frac{1}{2}d,cr$ (by 66.) And bec. $d,s,cr : z,s,cr :: d,s,\angle s : z,s,\angle s$ (by 81, 2, 1.) th. (by 66) $t,\frac{1}{2}d,cr$ ($s,\frac{1}{2}d,cr \div cs$) : $t,\frac{1}{2}z,cr$ ($s,\frac{1}{2}z,cr \div cs$) :: $t,\frac{1}{2}d,\angle s : t,\frac{1}{2}z,\angle s$, Th. $cs,\frac{1}{2}z,cr : cs,\frac{1}{2}d,cr :: cr,\frac{1}{2}\angle : t,\frac{1}{2}z,\angle s$. This Demonstration, with several other New and Valuable things of this Nature, I had from the Excellent Geometer Mr. *Halley*, whose Freedom in Communicating, and Readiness in Assisting, I shall always own with the highest Gratitude.

4. The two *Cases* excepted, *viz.*

1. *Three Sides being given; the Angles reqd.* Let x,z be the *Legs* of the ∠ sought, and b the *Base*, or Side op.

Then $\overline{l,x + l,z + L,\frac{1}{2}\overline{b + x \backsim z} + L,\frac{1}{2}\overline{b \backsim x \backsim z}} \div 2 =$ *Log.* cs, or s, $\frac{1}{2}\angle$. For in $\triangle no\lambda$, (*Fig.* 1. *p.* 280.) $\angle o = me = \frac{x}{}$;

$s,\angle n = s,\frac{1}{2}\overline{b + \overline{x + z}}$; $n\lambda = b \backsim \overline{x + z}$; $o\lambda$ *equiv.* $\chi m =$ Arc $\omega m =$ Suppl. $\angle v$; me ($\frac{1}{2}m\omega = s,\frac{1}{2}$Arc $m\omega$) $= cs,\frac{1}{2}\angle v$; Draw $eg \parallel \chi\omega$; $v\lambda$ ($s,v\lambda$) $= s,z$: Then $s,o : s,n :: \frac{1}{2}n\lambda : \frac{1}{2}o\lambda$, and $v\lambda : \frac{1}{2}o\lambda$ (:: $r : mg$) :: $r^2 : (r \times mg)\ me^2$. Th. $s,o \times v\lambda : \frac{1}{2}n\lambda \times s,n :: r^2 : me^2$, that is,

 $s,x \times s,z$

$$s,x \times s,z :: s,\overline{\tfrac{1}{2}b \backsim \overline{x+z}} \times s,\overline{\tfrac{1}{2}b + \overline{x+z}} :: r^2 : cs^2,\tfrac{1}{2}\angle v.$$

And after the like manner 'tis prov'd that

$$s,x \times s,z : s,\overline{\tfrac{1}{2}b \backsim \overline{x \backsim z}} \times s,\overline{\tfrac{1}{2}b + \overline{x \backsim z}} :: r^2 : s^2,\tfrac{1}{2}\angle v.$$

2. *Three Angles being given; the Sides required.* Instead of the greatest $\angle$, take its Supplement, and call the $\angle$s Sides, and the Sides $\angle$s; then do as in the preceding *Case.*

82. The Principles of Mechanics.

1. *Velocity* (v) or that Affection of Motion, whereby a Body runs a given *Space* (s) in a given *Time* (t) is the Ratio of the *Space* to the *Time*, *i. e.* $v = s \div t$; Th. $vt = s$, and $VTs = vtS$. Th. if $V : v :: T : t$, then $S \times t^2 = s \times T^2$, or $S \times v^2 = s \times V^2$.

2. *Moment* (m) or that which conduces to the effecting of *Motion*, is compounded of the *Velocity* (v) and *Quantity of Matter* or *Weight* (w) *i. e.* $m = vw$; Th. $mVW = Mvw$. And bec. $V = S \div T = M \div W$, th. $TM : tm :: WS : ws$.

That *Motion* is said to be *Equable*, which runs over all the Parts of Space with the same Velocity; but *Accelerated*, or *Retarded* when its *Velocity* is continually *Augmented*, or *Diminish'd*. The *Innate* or *Natural Force* of a Body is that by which it endeavours to persevere in its State of Rest, or uniform direct Motion. An *Impress'd Force* is an Action exercis'd on a Body to change its State of Rest, or Motion. *Centripetal Force* is that by which a Body is Impell'd, or Attracted towards some Point as a Centre. *Centrifugal Force* is that by which a Body endeavours to recede from its Centre.

3. *All Bodies will continue in their State of Rest, or uniform direct Motion, unless they are compell'd to alter that State by some Force impress'd upon them.*

4. *The Change of Motion is ever proportional to, and its Direction is in the same right Line with the Impress'd Force.*

5. *The Actions of two Bodies upon one another are always equal, and have contrary Directions.*

6. *Hence, if a Body* g (in Fig.6.) *be impell'd by two different Forces* (F, f) *to move* (in gδ, gλ) *with an uniform Velocity, it will describe the Diagonal* (gh) *of a Parallelogr.* in

in the fame time as it would defcribe the Sides (gδ,gλ) *by* F, f, *feparately*, For, becaufe the Force *F, f* (in gδ, gλ) hinders not the Velocity of the Force *f, F* from carrying the Body (g) to a Line (λh,δh) paral. to the Direction of *F,f*, in the fame time; whether the Force *F,f* be imprefs'd or not; th. (g) will be found in each of the parals. *i. e.* in their Interfection.

7. *Whence we have the Method of Compounding and Refolving any given Directions.* For any *Motion* may be confider'd as *Compounded* of others, and th. may be *Refolved* into them.

8. *If a Body be held immoveable by two equal Forces acting with contrary Directions*, fince either of thefe Forces may be refolv'd into two others, theref. *'tis the fame as if the Body was held by three different Forces; And thefe three Forces are one to the other, as Lines drawn parallel to the refpective Directions, and terminated at their mutual Concourfe.*

9. *Alfo the Proportion of an Oblique Force to move a Body, is to that of the fame Force coming with a Perpendicular Direction, as the Sine of the* ∠ *of Incidence, is to the Radius.* For the Oblique Force is compounded of two Forces, the one ||, and the other ⊥ to the Surface of the Obftacle, which is only affected by the latter or *Sine of Incidence*, the Oblique Force being made *Radius*.

10. *Hence the Forces of a Fluid Medium on a Plane cutting the Direction of its Motion at different Inclinations, are as the Squares of the Sines of the* ∠*s of Incidence.* For the Force of a Particle is as the Sine of Incidence; And the Number of Particles, that ftrike in equal time, is as the Sines of Incidence; Th. the Forces of all the Particles are as the Squares of the Sines of the ∠s of Incidence.

11. *If the Velocity of a Medium be different, the Forces on a Plane cutting that Medium with the fame Inclination, are as the Squares of the Velocity.* For the Force of each Particle is as its Velocity; And the Number of Particles, that ftrike in equal time, is as their Velocities: Th. the Forces are as the Squares of the Velocity.

12. The Force of the Water upon the Rudder of a Ship in Motion is as the Sq. of the Sine of the Inclination of the Rudder to the Keel (by 11); And the Force of the Rudder upon the Keel is as the Co-Sine into the Sq.

of the Sine of that Inclination, *i. e.* as $\overline{r^2-s^2}|^{\frac{1}{2}} \times s^2$; which if a *Maximum*, then $-s^3\dot{s} \div \overline{r^2-s^2}|^{\frac{1}{2}} + 2s\dot{s} \times \overline{r^2-s^2}|^{\frac{1}{2}} = 0$, Th. $s = \sqrt{\frac{2}{3}r^2}$ (= 54°,44′,08″ near) = *Sine of that Angle which the Rudder, in the most advantageous Position, should make with the Keel.*

13. *And the Force* (f) *of a Fluid Medium upon a* ⊾ △ *moving according to the Direction of its Base* (b) *is to the Force* (F) *on the Circumscr. Parallelogr. as the Sq. of the Perp.* (p) *to the Sq. of the Hypotenuse* (h) *i. e.* f, △ : F, ▭ :: $p^2 : h^2$. And the Force, striking with the Velocity (v) upon the Surf. descr. by p, taken infinitely small, about an Axe, at the Distance y, is as vpy (for py is as that Surf.) Th. $h^2 : p^2 :: vpy : vp^3y \div h^2$ the Force on the Surf. descr. by h, at the same time.

Th. if H, h be any two adjoyning Particles of a Curve, $yp^3 \div h^2 + YP^3 \div H^2$ or $yp^3 \div \overline{p^2+b^2} + YP^3 \div \overline{P^2+B^2}$ is the Force of the Fluid on the Surface generated by the Rotation of $H + h$; which if a *Minimum*, then (B, b being variable) $2b\dot{b}yp^3 \div h^4 = -2B\dot{B}YP^3 \div H^4$ or (bec. $B + b$ is constant, th. $\dot{b} = -\dot{B}$) $BYP^3 \div H^4 = byp^3 \div h^4$, or $\dot{x}y\dot{y}^3 \div \dot{z}^4$ or (sup. $P = p$) $\dot{x}y \div \dot{z}^4$ = an *Invariable: The Property of the Curve that generates the Surface of a Solid, which moving in a Fluid Medium, according to the Direction of its Axis of Rotation, shall meet with less Resistance than any other Solid generated by a Curve described to that Axe, and passing thro' the Extremity of the given Ordinate* y.

And drawing ϖ ⊥ to the Axe, and τ ∥ to the Tangent, the ⊾ △ made by ϖ, τ, β is *Sim.* to △ made by p, h, b, th. $4\varpi^2\beta\, (4\varpi^2 \times \varpi\dot{x} \div \dot{y}) : \tau^3\, (\varpi^3\dot{z}^3 \div \dot{y}^3) :: \tau\, (\varpi\dot{z} \div \dot{y}) : y$.

14. *Let the unequal Radii* CL, Cl (Fig. in p. 289.) *sustain the Weights* P, W, *by the Cords* LP, lW; *Required their Forces to move the Wheel:* Sup. (ϖ,w) *CD*,*Cd* ⊥ to the Directions of *P*,*W*, with *CD* descr. a Circumf. cutting the Direction of W in *g*; Then (since a Body hanging freely by any Point, makes that Point as heavy as if

if it exifted there) the Weight W is as gb, and its Force to turn the Wheel is as $g\delta$, And if $W : P :: (gb : g\delta :: gC$ or) $CD : Cd$, *i. e.* if $P \times \varpi = W \times \omega$, or when $P \times sp. P = W \times sp. W$, Then contrary *Forces* and *Refiftances* will fuftain one another; Th. if P be greater than $\overline{\omega \div \varpi} \times W$ or $\overline{sp. W \div sp. P} \times W$, the *Power* will overcome the *Refiftance* or *Weight*, with a Force equal to the Excefs. *Hence any given Weight or Refiftance, may be moved or overpower'd by any given Force: Confequently, the greateft conceivable Weight may be moved by the leaft conceivable Power.* And,

15. In the *Leaver*, or *Steel-yard*; when $P \times \varpi = W \times \omega$, the Power will fuftain the Weight or the Weights will Equiponderate. Theref. in the *Ballance*, where $\varpi = \omega$, the Weights muft be equal.

This Property of the *Leaver* explains the Powers of *Oars* in *Rowing*; of *Iron Crows* in moving or lifting Weights; of the *Hammer* in drawing out Nails, &c. alfo of *Pincers*, *Shears*, &c. which confift of double *Leavers* bearing on a Common *Fulcrum*.

If two *Fulciments* (x, z) fuftain a *Leaver* Horizontally, and c be the Centre of grav. of the Burden; Then (with refpect to the fix'd Points x, z) $p, x : W :: zc : xz$, & $p, z : W :: xc : xz$, Th. $p, x : p, z :: zc : xc$; And x, or z will fuftain $zc \div zx$, or $xc \div zx$ of W. Hence a Weight carried between two Men may be placed on the *Leaver* in any given Proportion; And fo two Horfes drawing any Weight, the Refiftance may be divided according to the Strength of the Horfes.

If two *Fulciments* (x, z) fuftain a *Leaver* oblique to the Horizon, and a, e its Interfections with the $\perp$s to the Horizon, paffing thro' the C. of Gravity of the Weight (W) fuftain'd *Below*, or *Above* that *Leaver*; Then x, z, will bear $\overline{az} \div \overline{xz}$, $\overline{ax} \div \overline{xz}$ part, or $\overline{ez} \div \overline{xz}$, $\overline{ex} \div \overline{xz}$ part of W. And if the Centre of Grav. be *Below*, or *Above* the *Leaver*, the *Superior*, or *Inferior* Fulciment will bear fo much the greater part of W, as the *Leaver* is more inclin'd.

If a Weight (W) be placed on Three Beams $(\alpha p, \beta p, \gamma p)$ joyn'd in p, and fupported by Three Props α, β, γ; fuppofe each Beam a Line, and produced thro' p, to meet the Sides of a Δ (whofe angular Points are at α, β, γ) in a, b, c; Then α, β, γ, will bear $pa \div \alpha a$, $pb \div \beta b$, $pc \div \gamma c$ part of W.

Hence,

Hence, in *Building*, where the *Joysts* or *Girders* are too short, they may be placed so as to sustain each other; and by the Combination of such, any *Flooring*, or the like, may be compleated, and the Weight sustain'd at each Contignation may readily be computed.

Hence also, the Strength of Beams fix'd at one End is easily found: And that a *Parabolic*, or a *Prismatic Parabolic* Beam, has the same Resistance in all its parts; Theref. $\frac{2}{3}$, or $\frac{1}{3}$ of the Timber, will in this Case, do the same Service: Which in *Civil* and *Naval Architecture* is is of no small Importance.

16. In the *Wheel*; when *Power* × *Rad. of the Wheel* = *Weight* × *Rad. of the Windlace*, the *Power* will sustain the *Weight*. Whence the Force of the *Capstan*, the *Crane*, &c. as also of all *Engines* with *Tooth'd Wheels*, may be accounted for.

17. In the *Pulley*; (let n = Number of Falls or Cords at the Block hook'd to the Weight) when P : W : : (*Sp.* W : *Sp.* P : :) 1 : n, or when P = $\frac{1}{n}$ W, the *Power* will sustain the *Weight*.

18. In the *Wedge*; the Weight w lying on the two Oblique Planes at, $a\tau$ ($\perp L\tau$) in Fig. p. 289. may be consider'd as a *Wedge*, where f, w on $a\tau$: f, *Hammer* (according to the Line wx) on that Plane : : $tx : wx$, and f, w on $a\tau$: f, w on at : : $tx : wt$. And when *Force* × *Altitude of the Wedge* = *Resistance* × *Thickness*, the *Power* will be Equivalent to the *Resistance*. This accounts for the Force of *Hatchets*, *Chisels*, and all *Edg'd* or *Pointed Instruments*, as also *Saws*, *Files*, &c.

19. In the *Screw*, which may be taken as a *Wedge* impell'd by a *Leaver*; When *the Power that turns the Screw* × *Circumf. descr. by it* = *Resistance of the Obstacle* (to be press'd or rais'd) × *Distance between the Thrids of the Screw*, then the *Power* will be Equivalent to the *Resistance*.

20. Let a Weight ($w = P$) be sustain'd partly by a Power at L, and partly by the Inclin'd Plane ab; draw wx, and $wt \perp$ *Horizon*, and *Plane*; then the descending Force of w may be express'd by wx, which may be resolv'd into the Forces wt, tx: Th. f, w descending along the Incl. Pl. ab : f, w along the Perp. Pl. as : : ($tx : wx$: : s,$\angle xwt$: s,$\angle twy$: :) s,$\angle$ of *Inclination* : cs,$\angle$ of *Traction*.

21. Th. if a Power at L sustain a Weight (w) upon an Inclin'd Plane, by a Direction (Lw) which, passing thro' the

the C. of Grav. of w, meets the Plane; Then *Power : Weight* :: s,∠ *Inclination* : cs,∠ *Traction*. If $Lw \parallel ab$, Then (bec. Δwtx Sim. Δbsa Sim. ΔsFa, if $sF \perp ab$) *Power : Weight* :: ($tx : xw$ ::) $as : ab$ :: s, *Inclinat.* : *Radius*. Th. at 30 *Deg. Incl.* the *Power* must be double of that which is necessary to draw a *Weight* upon an Horizontal Plane.

If $Lw \parallel sb$, then *Power : Weight* (:: $tx : wx$) :: $as : sb$ :: s, *Incl.* : cs, *Incl.* And *Power* at L, when $Lw \parallel ab$: *Power* at L, when $Lw \parallel sb$ :: $sb : ab$.

22. *If* as, ab, *be two Planes, the one* ⊥, *the other Oblique to the Horizon, let* sF ⊥ ab; *Then the Initial Forces, or Celerities of Descents, acquir'd in the same time, are* :: (sa : aF ::) ab : sa, *i. e. are reciprocally as the Lengths of the Planes.*

23. *And in equal Times, Sp. run in* ab : *Sp. run in* as :: (*vel. in* ab : *vel. in* as :: as : ab) :: aF : as, *i. e. The Weights shall descend thro'* aF, *and* as *in equal Times. And* (*drawing* μn ∥ sF) *the Spaces* μs, nF *shall be pass'd over in the same Time.*

Th. if the Diameter of a Circle be ⊥ *to the Horizon, a Body shall descend in the same Time thro' any Chord whatsoever conterminous to that Diameter.*

24. *Times in* ab, as, *are* :: (*Times in* ab, aF :: $\sqrt{ab} : \sqrt{aF}$ ::) ab : as, *i. e. Times of Descents thro' Planes equally high are directly as the Lengths of the Planes. Th.* (*if* μF ∥ sb,) *Times in* Fb, μs :: Fb, μs.

25. *And since* v, *in* s : v, *in* F (:: as : aF :: $\sqrt{ab} : \sqrt{aF}$) :: v, *in* b : v, *in* F, Th. v, *in* s $=$ v, *in* b, *i. e. The Velocities acquir'd in falling thro' Planes equally high are equal.*

26. Or, let a Body descend by a Line (l) inclin'd to the Horizon, with a Velocity increasing as the Time (t.) Sup. b, c the *Sine* and *Co-Sine* of the *Inclination*, v the last Velocity; Then (l and c being variable) $\dot{l} = c\dot{c} \div l$, but v is as $\sqrt{b}$, th. t, in $\dot{l}$ is as $c\dot{c} \div \sqrt{b} \times l$ or $c\dot{c} \div \sqrt{b} \times \sqrt{b^2 + c^2}$, th, t, in l is as $\sqrt{b^2+c^2} \div \sqrt{b}$ or $l \div \sqrt{b}$, th. t^2 is $l^2 \div b$, and $HT^2 l^2 = bt^2 L^2$, also (bec. $vt = l$) $HTlv = btLV$. Hence,

1. $T : t :: L \times \sqrt{b} : l \times \sqrt{H}$; And $V : v :: lHT : lbt$.
2. If $L : l :: H : b$, Then $T : t :: \sqrt{L} : \sqrt{l} :: V : v$.
3. If $H = b$, Then $T : t :: L : l$; And $V = v$.

4. If

4. If $L = l$, Then $T : t :: \sqrt{b} : \sqrt{H}$, And $V : v :: t : T$.

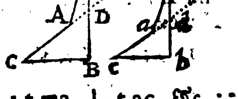
5. If $L^2 b = l^2 H$, Then $T = t$, And $V : v :: L : l$. Th. the Times of Descents thro' the Chords (L, l) of Arcs (whose vers'd Sines are H, b) are equal. For, in a Circle $L^2 : l^2 :: H : b$. (by 65.3.)

27. If a Body descends thro' how many soever contiguous Planes, however inclin'd, it shall acquire the same Velocity in the lowest Point, as if it had descended by the Perpendicular. Th. a Body descending by the Circumf. of a Circle, or any Curve Line, shall acquire in the Lowest Point, that Velocity which it would get by a Perp. fall from the same Height. And, bec. a Body thrown upwards with that Velocity which it got last by a Perp. fall, does ascend to the same Height, a Body if carried upwards (with the Velocity acquir'd in the Lowest Point, in descending by any Curved Surface) by the same, or any other Surface, however inclin'd, shall ascend to the same Height from whence it came, and have, in Points equally high, the same Velocity.

28. *The Velocities of a Pendulous Body (at* s*) describing different Arcs* (sf, sF) *are in the lowest Point* (s) *as the Chords of those Arcs.* For (drawing $cf, \mu F \perp as$,) v, in s are as ($\sqrt{cs} : \sqrt{\mu s} :: \sqrt{csa} : \sqrt{\mu sa} ::$) $sf : sF$.

29. *And* (since the Direction of the Impetus of a Body descending by a Curve Line is Horizontal in the lowest Point) *if a Pendulous Body be struck in the lowest Point* (s) *by an Horizontal Force equal to that in falling thro'* μs, *it shall ascend from thence to the same Height.* Since the smallest Arcs coincide with their Chords, which in the lowest Point are *Isocronal, Th. the smallest unequal Oscillations are perform'd in the same Time. But greater Arcs are not as their Chords, Th. Longer Pendulums, that describe fewer Degrees, err less; And greater Arcs take up a little more time than shorter ones do.*

30. *Let* MAC, mac *be several contiguous Planes, equally inclin'd to one another, and to the Horizon, and of Proportional Height; Then the Times in which a Body runs thro' those Planes, shall be in a Subduplicate Ratio of their Altitudes* MB, mb. For ($t,MA : t,ma :: \sqrt{MA} : \sqrt{ma} :: \sqrt{FA} : \sqrt{fa} :: \sqrt{FC} - \sqrt{FA} : \sqrt{fc} - \sqrt{fa} :: t,AC : t,ac ::$) $t,MA + t,AC$, &c. : $t,ma + t,ac$, &c. :: $\sqrt{MB} : \sqrt{mb}$. Th.

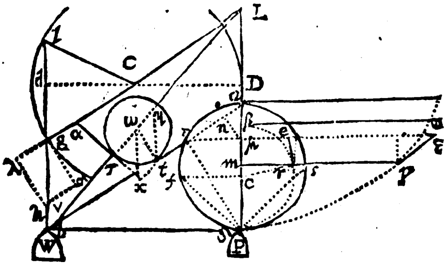


Th. in two Pendulums describing like Arcs (if $L,l =$ *Lengths*, $T,t =$ *Times*, $N,n =$ *Number* of *Vibrations* in the same Time) $T : t :: \sqrt{L} : \sqrt{l}$; *And* (since $NT = nt$) $N : n$ $(:: t : T) :: \sqrt{l} : \sqrt{L}$, or $N^2 : n^2 :: l : L$. Th. if $L =$ 39,125 *Inch.* and $N = 60$ *Vibrations* in 1'' of *Time*; Reqd. l, so as to make n or sup. 100 *Vibrations* in that *Time*;

Then $l = LN^2 \div n^2 = \overline{39{,}125 \times 60^2} \div 100^2$. Also if l $= 20$ *Inches*, and n the *Vibrations* in 1'' be requir'd, Then

$n = \overline{N^2 L \div l}|^{\frac{1}{2}} = \overline{60^2 \times 39{,}125 \div 20}|^{\frac{1}{2}}$.

31. *The Times, in which a Weight descends from any Point* (ϖ) *of a Cycloid, are equal among themselves: And have to the Time of a Perpendicular fall thro'* (as) *the Axis of the Cycloid, the Ratio of ½ the Circumf. of a Circle to the Diameter.*



For, let $\varpi\beta \perp$ sa, *and any Rt. Line* pm $\parallel \varpi\beta$, *cutting the ½ Circumf. on* sa, sβ *in* ϱ, r; *let* pε *and* re *touch the Cycloid and Circle in* p *and* r, *draw* $\varepsilon\mu \parallel$ pm, *and suppose 'em infinitely near, Then will* re, *and* pε *coincide with their respective Arcs*; *But* $p\varepsilon : m\mu$ $(:: \varrho s : sm :: \alpha\ (sa) : s\varrho$ (*by* 65.2.)) $:: \sqrt{\alpha} : \sqrt{sm}$, *th.* $p\varepsilon = m\mu \times \sqrt{\alpha} \div \sqrt{sm}$, *and vel. in*

p *or* m *is as* $\sqrt{m\beta}$, *th.* t, *in* pε *is* $\left(\frac{p\varepsilon}{\sqrt{m\beta}} = \frac{m\mu \times \sqrt{\alpha}}{\sqrt{sm\beta}} = \right.$

$\left. \frac{m\mu}{mr}\sqrt{\alpha} = \frac{re}{cr}\sqrt{\alpha} = \frac{re}{\frac{1}{2}d} \times \frac{\alpha}{\sqrt{\alpha}} = \frac{re}{d} \times \frac{\alpha}{\frac{1}{2}\sqrt{\alpha}} = \right) \frac{re}{d} \times t,$

in α, Th. t, in all the pε's (*i. e.* in the *Cycloidal Arc* ϖs)

is $=$ *to* $\overline{\textit{all the } re\textit{'s}, \textit{ or } \tfrac{1}{2}c}$ $\div\ d \times t$, *in* α, *And* t, *in the Cycloid* $\times\ d = t$, *in the Axe* $\times\ \frac{1}{2}c$.

32. Hence, $c : d :: t$, *of* 1 *Oscillation* : t, *of Descent in* $\frac{1}{2}$ *the Length* (l) *of the Pendulum*. For l = twice the Axe (by 72.4)

Put λ for the Length of a *Pendulum*, that measures 1″ of Time in each *Oscillation*; Then $c : d :: v(t, 1'') : dv \div c$ = *Time* (τ) in $\frac{1}{2}\lambda$; And ($\tau^2 : v^2 ::$) $d^2 : c^2 : \frac{1}{2}\lambda : b$ = Height fallen from in 1″ of Time. But 'tis found, by means of *Clocks* adjusted by the Heavens, that λ=39,125 Inches *London* Measure, Th. b ($= \overline{\frac{1}{2}\lambda \times c^2} \div 12\, d^2$) = 16,0895 *Feet*, or 16 *F.* and about 1 *Inch*.

The Resistance of the Medium does somewhat vary the Time of Descent in a *Cycloid*; But in a Medium that does not Resist, the shorter Oscillations in a *Cycloid* are nearly *Isocronal*. And the greater the *Funipendulous* Body is, the less does the Medium Resist it: But then the greater is the Distance of the *Centre of Gravity* from the *Centre of Oscillation* (*i. e.* the Point whose Distance from the *C. of Suspension* is the Length of a *Simple Pendulum*, whose Vibrations shall be *Isocronal* to those of the given Magnitude, or that Point wherein all the Figure is supposed to be contracted, with the Forces, while it vibrates.) A Right Line parallel to the Horizon, about which the *Oscillation* is made, is called the *Axis of Oscillation*. And every right Line, or Plane passing thro' the *Centre of Gravity* is an *Axis*, or *Plane of Equilibrium*.

33. *Given two Weights* (W,w) *and the Distance* (Dd) *between their Centres of Gravity* D,d: *Reqd.* (g) *their Common Centre of Gravity, or that Point in which all their Forces unite, or where, if they be joyntly suspended, they'll produce the same Effect as they did separately.* Let ($W : w :: dg :: Dg$, i. e.) $W + w : W :: Dd : Dg$, or $W + w : w :: Dd : dg$. Th. if there be any given Weights A,B, C, &c. and the Distance between their Centres of Gravity; Then (g) the Common Centre of Gravity of 'em all is easily found: For let α be the Centre com. to A and B, also β the Centre of A + B and C, &c. And if any Weight A,B,C, &c. be apply'd to any Points of a Line suspended at a given Point, and distant from it by a,b,c, &c. Then will (d) the Distance of their Common *Centre of Gravity* (g) from the Point of Suspension be = to $\overline{Aa + Bb + Cc, \&c.} \div \overline{A + B + C, \&c.}$ or $\overline{M + m + m, \&c.} \div \overline{W + w + w, \&c.}$ Also, if there be given several Weights A,B,C,

A,B,C, &c. (in the same or different Planes) whose Distance from the Axis of Oscillation call a,b,c, &c. and the Distance of their Common Centre of Grav. from the same is d; Then the Distance (δ) of the Centre of Oscillation from the Point of Suspension (or the Length of a Simple *Pendulum* that shall move as fast as a *Pendulum* composed of all the Weights) is equal to

$\overline{Aa^2+Bb^2+Cc^2, \&c.} \div \overline{A+B+C, \&c.} \times d$, that is,

$\overline{F+f+f, \&c.} \div \overline{M+m+m, \&c.}$ Th. if the Weights are equal, and their Number n, then $\delta = a^2+b^2+c^2, \&c. \div nd$.

34. In Quantities that are *Suspended* to, or do *Oscillate* about an Axis (A); Let d,δ be the Distance of the *Centre of Gravity*, or *of Oscillation* from (A) the *Axis of Suspension*, or *of Oscillation*: x the *Abscissa* of a Curve whose *Ordinates* (y) are parallel among themselves, and right to the Diameter; λ the *Length* of the Curve; α the *Area* adjacent to the Abscissa or its Parallel, and Flowing $\perp$ to A; ς the *Surface* of a Solid generated by the Rotation of a Plane adjacent to the Abscissa or its Parallel, about any Side of the Curve's circumscr. Pgr. and Flowing $\perp$ to A; χ the Content of such a Solid; p the greatest Perp. from the Curve to A; let m represent either λ,α,ς, or χ; and φ the *Fluent* of any *Fluxion*: Then,

1. *Where* A *is a Tangent to the Vertex of the Curve*; $d = \varphi,\dot{m}x \div m$; $\delta = \varphi,\dot{m}x^2 \div \varphi,\dot{m}x$.

2. *Where* A *is the Base of the Curve*; $d = pm - \varphi,\dot{m}x \div m$; $\delta = \varphi, \overline{p-x}|^2 \dot{m} \div \varphi, \overline{p-x}\,\dot{m}$.

3. *Where* A *is the Abscissa*; $d = \varphi, \dot{m}y \div m$; $\delta = \varphi,\dot{m}y^2 \div \varphi,\dot{m}y$.

4. *Where* A *is parallel to the Abscissa*; $d = pm - \varphi,\dot{m}y \div m$; $\delta = \varphi, \overline{p-y}|^2 \dot{m} \div \varphi, \overline{p-y}\dot{m}$. *Examples* in Case 1. If $y = x^n$, then $\dot{\alpha} = (y\dot{x})\dot{x}x^n$, th. $\alpha = x^{n+1} \div n+1$,

Th. $d = \left(\frac{\varphi,\dot{\alpha}x}{\alpha}\right)\frac{n+1}{n+2}x$; And $\delta = \left(\frac{\varphi,\dot{\alpha}x^2}{\varphi,\dot{\alpha}x}\right)\frac{n+2}{n+3}x$.

 Also,

Alſo, bec. $\dot{\chi} = \dot{x}x^{2n} \times \overline{c \div r}$, & $\chi = cx^{2n+1} \div \overline{r \times \overline{2n+1}}$

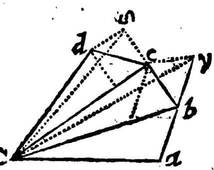
Th. $d = (\frac{\varphi, \dot{\chi}x}{\chi})\frac{2n+1}{2n+2}x$; And $\delta = (\frac{\varphi, \dot{\chi}x^2}{\varphi, \dot{\chi}x})\frac{2n+2}{2n+1}x$

Note, A Surface, or Solid generated by the Uniform Rotation of a Line, or Surface about an Axe, is equal to a Surface, or Solid whoſe Baſe is that given Line, or Surface, and whoſe Altitude is = to the Periphery deſcr. by its Centre of Gravity.

In a Cylinder (whoſe Height = h, Rad. of its Baſe = r) ſuſpended by (P) the Extremity of the Axe, The Diſtance (gc) from g (the Centre of Grav.) to c (that of Oſcillation) is = $\frac{1}{3}Pg + r^2 \div 2h$. And in a very ſmall Rod (P*E*) ſuſpended by the End P, $gc = \frac{1}{3}g$E. But if P*E* be taken as the Diameter of a *Sphere* ſuſpended by P, then $cg = \frac{2}{5}$ Pg. And if a *Sphere* be ſuſpended from another Point (ϖ,) then $cg = \frac{2}{5}$ Pg$\times \overline{Pg \div \varpi g}$, *which gives the Centre of Oſcillation in Order to adjuſt the Pendulum of a Clock.* So that if ϖg remain the ſame, then cg will be as Pg; Th. if $\varpi g = 39{,}125$ *Inches*, and Pg = 1, then $cg = {,}\frac{1}{100}$ nearly; Alſo if $\varpi g = 39{,}125$, and Pg = 3, then $cg = \frac{9}{100}$ nearly.

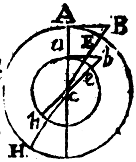
If (ϖg) the Diſtance between the Centres of Suſpenſion and Gravity be the ſame, then the *Greater* the *Sphere* is, the *Slower* will it Oſcillate; But if neither the Bigneſs of the *Sphere*, nor ϖg be alter'd, then, in *Reſiſting*, or *Non-reſiſting* Mediums, the *Lighter* the *Sphere* is, the *ſlower*, or *quicker* will it Oſcillate. And without having Reſpect to the *Centre of Oſcillation*, the Length of a *Pendulum* for determining any *Time*, or for any *Parallel of Latitude*, cannot be accurately found.

35. Suppoſe a Body in the 1ſt Moment of Time to run thro' *ab*, theref. the ſame Velocity continuing, it ſhall run in the 2d Moment thro' *bγ*; but in *b*, ſup. the Centripetal Force acting, draw *γc* ∥ *bC*, and = to the Length thro' which the Body is carried towards *C* in a Moment, then *bc* is run thro' in the 2d Moment; and the △*cCb* = △*γCb* = △*baC*: Likewiſe in the next Moment, will the Body deſcribe



ſcribe cd, making the $\triangle cCd = \triangle \gamma Cb$, and ſo on; *Whence, equal Areas are deſcrib'd in equal Times.* Let as well the Moments of Times, as the right Lines ab, bc, &c. be conceiv'd infinitely ſmall, then the Polygon $abcd$ will become a Curve Figure, from whoſe Tangent the Body is perpetually retracted by the *Vis-Centripeta*; *And the Areas deſcrib'd by the Radii drawn from the Centre* (C) *to the Body ſhall be Proportional to the Times of Deſcription.*

And if a Body move in a Curve Line, and by Rays drawn to the ſame Point deſcr. Areas Proportional to the Times, it is retracted from the Tangent to this Curve by a Centripetal Force tending to this Point.



36. Let C be the *Centrifugal* or *Centripetal Force*, T the *Periodical Time*, V the *Velocity*; Sup. AB a *Tangent*, BH a *Secant*, then $BE = AB^2 \div BH$ (by 64.19) But if AB be infinitely ſmall, then will $BE = AE^2 \div EH$; Th. (if AE, ae be Arcs deſcrib'd by two movable Bodies in the ſame Time; Then $C : c :: (AE^2 \div 2R : ae^2 \div 2r ::)\ V^2 r : v^2 R :: Rt^2 : rT^2$; Th. if $T^2 : t^2 :: R^2 : r^2$, then $CR = cr$; And $V = v$. If $T^2 : t^2 :: R^3 : r^3$, then $CR^2 = cr^2$; And $V \times \sqrt{R} = v \times \sqrt{r}$.

'Tis obſerv'd, *That the Primary Planets do deſcribe equal Areas about the Sun in equal Times*; *And that the Squares of their Periodic Times are as the Cubes of their Diſtances from it*; Th. 'tis the *Sun* that is the Centre of the *Planetary Syſtem.*

It follows alſo, that the Force of Gravity of the Heavenly Bodies is Reciprocally as the Squares of their Diſtances from the Centre. And $V : v :: r : \sqrt{Rr} :: \sqrt{Rr} : R$, *th. the nearer the Centre, the greater the Velocity.*

If 2 Bodies deſcribe equal Areas (CAE, cae) in equal Times, or if $CA \times AE = ca \times ae$, *i. e.* $VR = vr$, and $V^2R^2 = v^2r^2$) Then $C : c :: (V^2 \div R : v^2 \div r :: r^2 \div R : R^2 \div r ::)\ r^3 : R^3$.

37. The $\angle$s, A, a (which, a Body deſcr. about the Centre of Attraction) are as r^2 to R^2. For $V : v :: RA : ra$, th. $A : a$ ($:: Vr : vR$) :: (bec. if $T = t$, then $V : v :: r : R$) $r^2 : R^2$.

38. If v, V = *Velocity* acquired in the *Time* t, T, s = *Space* run, ſuppoſing, $t^n : T^n :: v^m : V^m$; S = *Space* run by an *Equable Motion* in the *Time* T, with *Velocity* V, (let

$m \div n = x$) Then $t = Tv^x \div V^x$, th. $\dot{t} = x\dot{v}\ Tv^{x-1}$ $\div V^x$, but $\dot{s} = v\dot{t} = x\dot{v}\ Tv^x \div V^x$, th. $s = x \div \overline{x+1}$ $\times Tv^{x+1} \div V^x =$ (putting V for v) $m \div \overline{m+n} \times TV$; And $S = VT$, Th. $s : S :: m : m+n$; Or if $T : t :: V : v$, then $s : S :: 1 : 2$.

Th. if S, and Σ, = *Space* run by an *Equable*, and an *Equably Accelerated* Motion in the *Time t*, and vt, Then $t^2 : v^2t^2 :: \frac{1}{2}S : \frac{1}{2}Sv^2$, and $S : (\frac{1}{2}Sv^2)\ \Sigma :: 2 : v^2$.

39. Let the Arc a be descr. by a Body revolving in a Circle with an *uniform Motion* in the *Time t*; and the Rt. Line b descr. by its Descent with *Accelerated Motion* in the *Time vt*; then (bec. $a : b :: 2 : v^2$,) $a = 2b \div v^2$; but the *Centrifugal Force (c)* in the time t is $a^2 \div 2r$ (by 36.) $= 4b^2 \div 2rv^4$, Therefore the Space run by the *Centrifugal Force (c)* in the *Time t* : sp. run by (*Gravity*) g in *Time vt* ($:: 4b^2 \div 2rv^4 .: b$) $:: 2b : rv^4$.

Th. if a Body (whose *Centrifugal Force* is c, *Gravitating Force g*) moves uniformly in a *Periphery* (p) whose *Rad.* is r, with a *Velocity* equal to that acquired by falling the *Height b*; Then shall $c : g :: 2b : r$, Th. if $b = \frac{1}{2}r$, $g = c$.

40. *If any two Pendulums, carried with a Conic Motion, descr. Peripheries* (P,p) *whose Radii are* R,r; *The Times* (T,t) *of Description shall be as the Square Roots of the Altitudes* (A,a) *of the Cones.* For $g : c :: a : r$, th. $c = (rg \div a)\ 2bg \div r$, and $b = (r^2 \div 2a)\ v^2$, th. $v = r \div \sqrt{2a}$, but ($p \div v$) $p \times \sqrt{2a} \div r =$ *Time t* in p; And ($p \times \sqrt{2a} \div r$: $P \times \sqrt{2A} \div R$ ::) $\sqrt{a} : \sqrt{A} :: t : T$. Th. if $A = a$, then $T = t$.

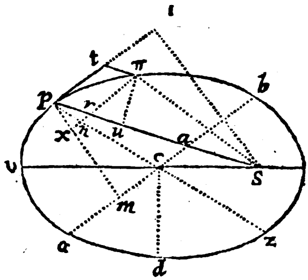
41. In a *Pendulum* carried by a Conic Motion; *Time in descr. the least Periphery : Time in falling an Alt. = twice the Length* (L) :: *Circumf. : Diameter*. For v, in 2A is $\sqrt{2A}$, and Time is $4A \div \sqrt{2A}$ or $2\sqrt{2A}$, And $2\sqrt{2A}$: $P \times \sqrt{2A} \div R :: 2R : P ::$ *Diam.* : *Circumf.* But (bec. R is sup. infinitely small) L = A, Th. &c. *Th. the Time in describing the smallest Circuit is = to the Time of the two smallest Oscillations of the Pendulum.*

42. If ($\overline{p \times \sqrt{2a} \div r}$) Time in defcr. $p = (2\sqrt{l})$ Time in falling Alt. $= l$; Then (bec. $4l = 2p^2 a \div r^2$) $a : l :: 2r^2 : p^2$, or s, *Incl.* : *Rad.* :: *Infcr. Sq.* : *Sq. Circumf.*

43. If a Pendulum defcends thro' a Quadrantal Arc, the *Vel.* in the loweft Point is = to that acquired in falling thro' r, th. $c = (2hg \div r) = 2g$, Th. the Force in the loweft Point is $= (2g + g)$ $3g$.

44. *The Spaces, which a Body impell'd by fome regular Force defcribes, are in the very Beginning of the Motion, as the Squares of the Times,* For if the *Times* be exprefs'd by ve, vf (*Fig. Art* 69,7.) and the *Celerities* in thofe *Times*, by $ep, f\pi$, then the *Spaces* fhall be as the *Areas* $vpe, v\pi f$, which are as ve^2 to vf^2 (by 69,7.)

45. If a Body in revolving about a Centre (s) defcribes a Curve Line $vp\pi$; Let pt be a Tangent, $t\pi$ || and $\pi u \perp ps$; Now time (T) is as $\Delta s\pi p$ (by 35) or as $\pi u \times sp$, and if T be given, the *Lineola* $t\pi$ is as C (the Centripetal Force,) but if C be given, $t\pi$ is as T^2, or as $\pi u^2 \times sp^2$, theref. univerfally, $t\pi$ is as $C \times \pi u^2 \times sp^2$, th. C is as $t\pi \div \overline{\pi u^2 \times sp^2}$.



Example. *If a Body revolve in an Ellipfe; to find the Law of the Centripetal Force tending to the Focus.* Draw the Diameter pz, and αb Conjugate to it, cutting ps in a; Let $pm \perp \alpha b$, and $\pi x \parallel pt$ cutting ps, pc, pm in r, n, x: Then $ap = cv$ (by 68. 4.) and $cb \times pm = cd \times cv$ (by 68.28.) alfo (bec. $\pi t : tp :: 1 : \infty$) $\pi r = \pi x$, and $zn = zp$; But $\pi u^2 : \pi r^2$ ($:: pm^2 : pa^2 :: pm^2 : cv^2$) $:: cd^2 : cb^2$, and $\pi r^2 : znp ::$ ($\pi n^2 : znp ::$) $cb^2 : pc^2$, th. $\pi u^2 : cd^2$ ($:: znp : cp^2 :: np : (cp^2 \div zp)\ \frac{1}{2}cp :: pr : \frac{1}{2}pa$) $:: t\pi : \frac{1}{2}cv$, th. $t\pi \div \pi u^2 = \frac{1}{2}cv \div cd^2$, th. C or $t\pi \div \overline{\pi u^2 \times sp^2}$ ($= \frac{1}{2}cv \div \overline{cd^2 \times sp^2}$) is as $1 \div sp^2$ (fince $\frac{1}{2}cv$, and cd^2 are ftanding Quantities) *That is, the Centripetal Force is reciprocally as the Square of the Diftance.*

And by a like reafoning, 'tis readily found, that if a Body move in an *Hyperbola*, or *Parabola*, *the Centripetal Force*

Force tending to the Focus, shall be reciprocally as the Square of the Distance. But if the *Centripetal Force* should tend to the Centre of the *Ellipse* or *Hyperbola* (which is no where found) it would be *directly* as the Distance.

Hence, if a Body move in a Rt. Line from any place, with any Velocity, and with a Centripetal Force that is reciprocally as the Square of the Distance, it shall move in some of the *Conic Sections*, whose *Focus* is in the *Centre of Forces.*

46. Let A,a be the *Areas* describ'd in different *Orbs* or *Ellipses* E,e, whose *Transv. Axes* are D,d, *Parameters* P,p, let T,t be the *Periodical Times.* Then if several Bodies revolve about a Common Centre, and that C is as $1 \div ps^2$, Then (bec. $P = (cd^2 \div \frac{1}{2}cv$, by 68.5) $\pi u^2 \div t\pi$ and in a given Time, $t\pi$ is as C (by 45) or as $1 \div sp^2$, th. P is as $(\pi u^2 \times sp^2 =) A^2$, That is, $P : p :: A^2 : a^2$, *Or the Parameters of the Orbs shall be as the Squares of the Areas describ'd in the same time.*

47. Th. if $T = t$, A is as $\sqrt{P}$, but if $P = p$, A is as T, Th. universally, $A : a :: \sqrt{P} \times T : \sqrt{p} \times t$, but $A : a :: \sqrt{P} \times \sqrt{D^3} : \sqrt{p} \times \sqrt{d^3}$ (by 68.26) th. $T^2 : t^2 :: D^3 : d^3$. *That is, the Squares of the Periodic Times are as the Cubes of the Transverse Axes, or mean Distances from the Centre of Attraction.*

48. The same things being supposed (as in 45,) draw $st (= \delta) \perp pt$ (the Tangent,) Then the Velocity (V) of the Body is as the little Arc $p\pi$ descr. in a given Time, *i. e.* as pt or πr, but $\pi r : \pi u :: sp : st$, th. pt is as $\pi u \times sp \div st$, *i. e.* V is as $(A \div \delta) \sqrt{P} \div \delta$ or $\sqrt{P \div \delta^2}$.

83. Of the Motion of Projects.

1. *A Body being Projected by any Force moves in the Curve of a* Parabola, *unless so far as the Resistance of the Medium hinders it.* Thus, if HK be $\perp$ to the Horizon, and any Rt. Lines $\mu v, mV$ be such, as that in the Time the Body by its *Projectile Motion* arrive to μ, m, it may fall, by its *Descending Motion*, the Length $\mu v, mV$; Then (by 82.1.) $\mu v : mV :: H\mu^2 : Hm^2$, or $Hk : HK :: kv^2 : KV^2$, which is the Property of the *Parabola.*

2. *The Horizontal Distances* (H,h) *of Projects, made with the same Velocities, at several Elevations* (E,e) *are as*

as the Sines of the double ∠s of Elevations. For (if s = *Sine*, ς = *Co-Sine*) x (RL) = $bs \div \varsigma$, and y (HL) = $br \div \varsigma$

th. (y^2) $\frac{pbs}{\varsigma} = \frac{b^2r^2}{\varsigma^2}$, th. $b = (\frac{ps\varsigma}{r^2} = \frac{\frac{1}{2}p}{r} \times \frac{2s\varsigma}{r}$ = (bec.

$\overline{2s\varsigma} \div r = s,2\angle$, by 15.65.) $s,2\angle \times \frac{1}{2}p \div r$, Th. $H : b ::$ $s,2\angle E : s,2\angle e$.

2. Hence, when $\angle e = 45°$, the Sine of its double is the greatest Sine; but the Ranges are as the Sines of the double ∠s, *Th. the greatest Random is at* 45° *Elevation.* And bec. at 45° *Elevat.* s, 2∠ = r, th. b (or g) = $\frac{1}{2}p$, i. e. *The greatest Random is equal to* ½ *the Parameter.* Also, the Ranges equally distant above and below 45° are equal.

3. *The Altitudes* (A,a) *of Projections, made with the same Velocity, at several Elevations* (E,e) *are as the Versed Sines of the double ∠s of Elevations.* For $\varsigma : s ::$ (b) $\frac{ps\varsigma}{r^2}$: (x) $\frac{ps^2}{r}$, And a (MV = Vm) = $\frac{1}{4}x = \frac{ps^2}{4r^2} = \frac{2s^2}{r} \times$

$\frac{\frac{1}{2}p}{4r}$ = (bec. $\frac{2s^2}{r}$ = v,2∠, by 15.65.) v,2∠, $\times \frac{\frac{1}{2}p}{4r}$ (= v,2∠ $\times \frac{p}{8r}$)

Th. $A : a ::$ v,2∠E : v,2∠e. And since $2r$ = greatest vers'd *Sine*, th. $r : 2r :: \frac{1}{8}p : \frac{1}{4}p$ = *greatest Altitude* = ½ the *greatest Random.*

4. *The Times* (T,t) *of the Flights of a Project thrown with the same Velocity at different Elevations* (E,e) *are as the Sines of the Elevations.* For let y (HL) represent the Time, then : r :: (b) $ps\varsigma \div r^2 : ps \div r = y$, th. $r : p :: s : y ::$ S : Y, *i. e.* T : t :: $s,\angle$ E : $s,\angle e$.

5. *Given* (b) *the Horizontal Distance of an Object,* (e) *the ∠ of Elevation,* (n) *the Ascent or Descent.* Reqd. *the greatest Random* ($g = \frac{1}{2}p$,) *and Velocity.*

1. cs, e : b :: s, e : $x \pm n$, and $b^2 + \overline{x \pm n}|^2 = y^2$. Or r : ſ,e :: b : y, and $y^2 \div x = p = 2g$.

2. Since x = Fall in the *Time* y, th. 193 *Inches* : x :: □ 1″ : $\frac{1}{193}x \times$ □ 1″ = *Sq.* of the *Time* in x, Th. *Velocity*

(S ÷ T) = $y \div \sqrt{\frac{1}{193}x} = \sqrt{px} \div \sqrt{\frac{1}{193}x} = \sqrt{193p}$.

6. Since the *Force* or *Velocity* of a Project, describing a Curve Line, is compounded of the Uniform Velocity in y, and of that Uniformly Accelerated in x; which will carry it but ½ the Space that it would run uniformly

ly with the last acquir'd Velocity in the same Time, (by 33. 82.) Therefore, the Velocity in any Point (P) of the Curve is to the Velocity impress'd in H, as the *Tangent* to the *Ordinate*; or as the Lengths of *Tangents*, to the Points P, and H, intercepted between the *Diameters* to those Points; or as the *Secants* of the $\angle$s made by those *Tangents* produc'd, and the Horizontal Line. Th. the *Least Force* is at the Vertex of the Curve, or at the Height of $\frac{1}{4}p$. And at equal Distances from the Vertex, the Forces are equal. Also, *Least Force : Impress'd Force* at $H :: r : s, \angle e$.

7. *Given*, $\angle e, g\ (\frac{1}{2}p), n$, *and* $\angle a$; *Required* b.

Bec. $\overline{ys} \div r = x \pm n$, th. $x = \overline{ys \mp rn} \div r = y^2 \div p$, Th. $y^2 - spy = \mp prn$; And $r : y :: cs, e : b$.

8. *Given*, $b, n, g\ (\frac{1}{2}p)$; *Required* E, e, (whose *Tangent* call t.) Since, $\overline{bt \div r}.\ (b) = x \pm n$, th. $b \mp n = x$; but $\overline{b \mp n} \times p\ \ (= xp = y^2) = b^2 + b^2$, and $b^2 = bp \mp np - b^2$. Therefore, $\overline{t \div r} = \overline{p \div 2b} \mp (f)\ \overline{\overline{p^2 \mp 4pn \div 4b^2} - 1}\,|^{\frac{1}{2}}$ or $t = r \times \overline{\frac{1}{2}p \mp \sqrt{\frac{1}{4}p^2 \mp pn - b^2}} \div b$.

Th. $b^2 : \frac{1}{2}p :: \frac{1}{2}p \mp 2n : \overline{p^2 \mp 4pn} \div 4b^2 =$ to a s^2, whose *Tangent* is f, And $f \pm \overline{\frac{1}{2}p \div b} = t, E, e$, *sought*.

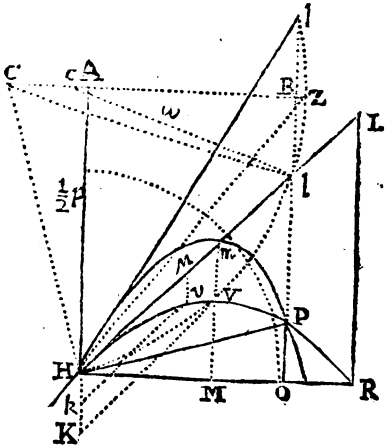
1. Th. if $f > \frac{1}{2}p \div b$, or if $pn > b^2$, then, in *Descents*, the Direction of the *lower Elevation* will be below the *Horizon*. But if $pn = b^2$, the Direction will be *Horizontal*, and $pr \div b =$ *Tangent* of the *upper Elevation*. And if $b^2 \pm pn > \frac{1}{4}p^2$, the Object (in *Ascents*, or *Descents*) is beyond the reach of the Project thrown with that Velocity.

2. If $\frac{1}{4}p^2 = b^2 \pm pn$, in *Ascents*, or *Descents*, there can be but one Elevation (whose Tangent is $\overline{pr \div 2b}$) that will reach the Object. Th. $n = \frac{1}{4}p - \overline{b^2} \div p$, or $\overline{b^2 \div p} - \frac{1}{4}p$, which determines the utmost Height of any ject upon each given *Horizontal* Distance. And $\frac{1}{2}p = \overline{\sqrt{b^2 + n^2}} \pm n =$ *Horiz. Range* at 45° of a Project, thrown with the least Velocity capable to reach the

the Object; Or the Charge requifite (in Afcents, or Defcents) to hit a Mark is that which at 45° Elevation wou'd throw the Shot the Diftance of $\sqrt{b^2+n^2} \pm n$ upon the Plane of the Horizon.

3. Th. $b : r :: \overline{b^2 + n^2}|^{\frac{1}{2}} \pm n$ ($z \pm n$, or ſ ± t, *Incl.*) : t, the *Tangent* of an Elevation proper for a Gun fo charg'd, to ftrike the Object with the greateft Certainty and Advantage: But ſ ± t, = ct, or t, $\frac{1}{2}$ $\overline{90° - Incl.}$ (by 16. 65.) Th. the *utmoft Range*, on an inclined Plane, is made, when the Axis of the Piece makes = ∠s with the Perp. and Object; So that there needs no more, than to let the Axis of a Piece, duely charg'd, bifect the ∠ (α) between the Perpendicular and Object.

4. The *Geometrical* Conftruction of the laft *Problem* may be thus. *Let* OB (⊥ OH) = $\frac{1}{2}p$, *and* Bc (|| OH) = HP.



From c, *with* BP ($\frac{1}{2}p \pm n$) *defcribe an Arc, which (if the thing be poffible) will cut* OB *in* l, l; *Then will* Hl *be the Lines of Direction fought.* For (if BP = w, Bl = u, HP = z) $w^2 - z^2 = u^2$ = (bec. $u = w - x$) $w^2 - 2wx + x^2$, th. $z^2 + x^2$ = ($2wx$) $px \mp 2nx$; And y^2 = ($z^2 + x^2$, $\pm 2nx$

$\pm 2nx = px \mp 2nx \pm 2nx =)\ px$; *i. e.* $Hl^2 = p \times Pl$. th. the Curve will pass thro' P. Or bec. $u = (\overline{w^2 - z^2}|^{\frac{1}{2}}$ $= \overline{\overline{\frac{1}{2}p \mp n}|^2 - \overline{b^2 + n^2}}|^{\frac{1}{2}} =)\ \overline{\frac{1}{4}p^2 \pm pn - b^2}|^{\frac{1}{2}}$, therefore $Ol = \frac{1}{2}p \pm u$, and $b : r :: \frac{1}{2}p \pm u$: (t, *E*, *e*) $r \times \frac{1}{2}p \pm \overline{\frac{1}{4}p^2 \pm pn - b^2}|^{\frac{1}{2}} \div b$. Otherwise, *Let* CH $\perp$ HP, AH ($\perp$HO) $= \frac{1}{2}p$, and AC || HO; *From* C, *with* CH, *describe an Arc cutting* (*if possible*) OB *in* l, l; *Then will* Hl *be the Lines of Direction.* For since AC $= \overline{\frac{1}{2}pn} \div b\ (= c)$, theref. $CB^2 = c^2 \mp pn + b^2$, and Cl^2 $(CH^2) = c^2 + \frac{1}{4}p^2$, th. $Bl = \overline{\frac{1}{4}p^2 \pm pn - b^2}|^{\frac{1}{2}}$ And $b : r :: \frac{1}{2}p \pm u$: (t, *E*, *e*) $r \times \frac{1}{2}p \pm u \div b$. Hence the 5*th*, 7*th*, and 8*th* *Proposition* may be *Arithmetically* Solv'd thus,

1. ($\frac{1}{2}\overline{E+e} = 90° - \frac{1}{2}\alpha$, th.) $90° - \frac{1}{2}\alpha \backsim e = \frac{1}{2}\overline{E-e}$ and v,α (AZ) − v,$\overline{E-e}$ (BZ) $= q$ (AB), Then q : s,α (AH), $:: b : \frac{1}{2}p = g$.

2. Having found q, then $s,\alpha : q :: \frac{1}{2}p : b$.

3. Let $\frac{1}{2}p : b ::$ s,α (AH) : q (AB), and v, $\alpha - q$ (AZ − AB) = v, $\overline{E-e}$ (BZ), but $\frac{1}{2}\overline{E+e}$ $(= \frac{1}{2}\overline{180° - \alpha}) = 90° - \frac{1}{2}\alpha$; And $\frac{1}{2}\overline{E+e} \pm \frac{1}{2}\overline{E-e}$ $= E, e$, required.

And because of the *Air's Resistance*, the *Line of Projects* is not exactly *Parabolical*, but rather a kind of an *Hyperbola*; which, if consider'd, and apply'd to Practice, would render the Computation far more operose, and the very small Difference (as Experience shews in Heavy Shot) would, in a great Measure, lessen the Elegancy of the Demonstrations given by Accounting for it; since the former Rules are sufficiently exact and easie for Practice.

The Theory of the *Motion of Projects* is so perplex'd a Subject, and depends so much upon Physical Observations, that such Accuracy cannot be expected therein, as don't require some Allowances. But the Resistance of the Medium, and other Accidental Impediments, may be in some measure

measure Rectified, by supposing the Shot to move in a Right Line to a certain Distance (d) from the *Axis* of the Gun, and afterwards to describe the Curve of a *Parabola*.

Given, E,e; H,h: Reqd. p,d. Since $d+y$, and x are given (by *Trigonom.*) also $y = \sqrt{p} \times \sqrt{x}$ (by 36. 68), and $d =$ D; th. $\overline{D+r} \backsim \overline{d+y} = r \backsim y = \overline{\sqrt{X} \backsim \sqrt{x}} \times \sqrt{p}$, Th. $p =$ $\overline{\overline{D+r} \backsim \overline{d+y} \div \overline{\sqrt{X} \backsim \sqrt{x}}}\big|^{\frac{2}{1}}$. And $d = \overline{d+y} - \sqrt{\overline{px}}\ (y)$

Given $d, \frac{1}{2}p, Le, n$; *Reqd.* b. Since $r : d+y :: s : \overline{ds+ys} \div r\ (= x \pm n) = y^2 \div p \pm n$, th. $ry^2 - psy = pds \mp rnp$; (let $r : s :: p : q$, or $rq = sp$) and $y^2 - qy = dq \mp np$, th. y is given; And $r : d+y ::$ cs,e : b sought.

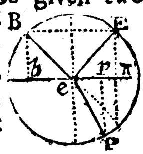
Given $d, b, \frac{1}{2}p, n$; *Reqd.* Le. Bec. $x \pm n = (y^2 \div p \pm n)\ \overline{y^2 \pm np} \div p$, th. $\overline{\overline{y^2 \pm np} \div p}\big|^2 + b^2 = \overline{y+d}\big|^2$, th. $y^4 \pm \overline{2np - p^2} \times y^2 - 2dp^2y = \overline{d^2 - b^2 - n^2} \times p^2$, th. y is given; And $d+y : r :: b :$ cs, Le sought.

84. *Of* Optics.

That Science, which accounts for the various Appearances of Objects, from the Reflexions, Refractions, and Inflexions of the Rays of *Light*, is called *Optics*.

Experience shews that the Light of the *Sun* consists of Rays that are differently *Reflexible* and *Refrangible*. The Angle comprehended between the Incident, Reflected, or Refracted Ray, and the Perpendicular to the Reflecting, or Refracting Surface is called the *Angle of Incidence*, *Reflexion*, or *Refraction*. And these three Angles lie in one and the same Plane. The *Refraction* out of a Rarer Medium into a Denser is made towards the Perpendicular; and the contrary.

And, supposing Nature's Method of working to be the most easie and expedite, if there be given two Points (B,P) in Mediums of different given Densities, and the Position of the Plane ($b\pi$) dividing those Mediums; Reqd. the Point (e) in that Plane, thro' which the Body passes by taking the shortest Time to move from B to P.



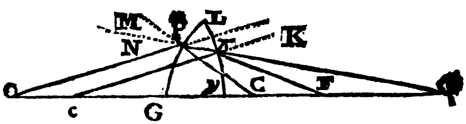
Let

Let $Be, eP = s, S$; $Bb, Pp = c, C$; $bp = a, be = z$, then $pe = a - z$; let $v, V =$ *Velocity* in s, S: Th. since t, in s : $t, S :: sV : Sv$, th. t, in $s + S$ is $\overline{c^2 + z^2}|^{\frac{1}{2}} \times V + \overline{C^2 + \overline{a - z}|^2}|^{\frac{1}{2}} \times v$, *which if a Minimum*, then $Vz\dot{z} \div s + \overline{vz\dot{z} - va\dot{z}} \div S = 0$. Th. $SVz = sv \times \overline{a - z}$, th. $SV : sv :: a - z : z$, Th. (supposing $s = S$) $v : V :: z : a - z$ $(:: m : n)$ i. e. *The Sines of the ∠ of Incidence, and of the Refracted ∠ are directly as the Velocities, or reciprocally as the Densities of the Mediums.*

But if a Body moving from *B* to *e* be *Reflected* to *E*, bec. it continues in the same Medium, th. $v = V$, conseq. $s : S :: z : a - z$, and ∠s at *b* and π are right, th. ∠Beb = ∠ $Ee\pi$, i. e. *The ∠ of Incidence is = ∠ of Reflexion.*

Homogeneal Rays, (or those of like Refrangibility) which flow from several Points of any Object, and fall almost Perpendicularly on any Reflecting, or Refracting Surface, shall afterwards Diverge, or Converge, to, or from so many other Points, or be Parallel to so many other Lines, either accurately or very near. The same happens if the Rays be reflected or refracted successively by two or more Surfaces. That Point to, or from which Rays Converge, or Diverge is called the *Focus.*

Given a double Convex Lens ($GL\gamma$) *whose Thickness is* $G\gamma$ ($= t$), *and* O *a Point or Object in the Axis of the Lens; the Ratio of Refraction being as* m *to* n : *Reqd. the Point* (F) *at which the Beams that are nearest the Axis of the Lens are collected.*



Let OG ($= OP$) $= d$, the Distance of the Object, CG ($= CP$) $= r$, and $c\gamma$ ($= c\pi$) $= \varrho$, the Radius of the Segment *towards* and *from* the Object : And let S, ∠ *Incid.* (OPM) : S, *Refr.* ∠ (NPM or $CP\varphi$) :: $m : n$; then, in very small ∠s, ∠ *Incid.* : *Refr.* ∠ :: $m : n$. Theref. $d : r ::$ ∠C : ∠O; th. $d + r$ is as the ∠ *Incid.* (OPM) and $m : n :: d + r : \overline{nd + nr} \div m$, which is as the *Refr.* ∠ ($CP\varphi$)

∠ (CPφ); But ∠ PCO — CPφ (or $d - \overline{nd + nr} \div m$) = ∠ PφO. And the ∠ φ : ∠ O :: PO : Pφ (= Gφ) = $mrd \div \overline{md - nd - nr}$, (that is, the Beams from O by the *first Refraction* will be collected in φ.) If *nr* be *greater*, or *equal* to *md* — *nd*, then the Beams after Refraction go *Diverging from*, or *Parallel to* the Axis, and the Point φ is *on the same side beyond P*, or *Infinitely distant*. Also (if $\delta = G\varphi - G\gamma = \varphi\gamma$ or $\gamma\varphi$) $\delta : \varrho$ :: ∠ c : ∠ φ; th. $\delta + \varrho$ is as the ∠ *Incid.* (Kπφ,) and $n : m :: \delta + \varrho : \overline{m\varrho + m\delta} \div n$, which is as the *Refr.* ∠ (KπF)

But ∠ KπF — KcF (or $\overline{\overline{m\varrho + m\delta} \div n} - \delta$) = ∠ cFπ. And the ∠ cFπ : ∠ KcF :: πc : πF = $n\varrho\delta \div \overline{m\delta - n\delta + m\varrho}$ = (if $a = n \div \overline{m - n}$) unto

$$\frac{madr\varrho - \overline{d\varrho - ar\varrho} \times tn}{m \times \overline{dr + d\varrho - ar\varrho} - \overline{md - nd - nr} \times t} = f =$$ (if *t* be rejected)

$adr\varrho \div \overline{dr + d\varrho - ar\varrho}$. Whence, any *four* of these *five* Quantities (d, r, ϱ, a, f) being given, the *fifth* is readily found. And,

In *Diverging Rays*, falling on a *double Convex*, or *Concave* (where it is $+d$, & $\pm r$, $\pm\varrho$) $\pm adr\varrho \div \overline{dr + d\varrho \mp ar\varrho} = \pm f$; But, in the *Convex*, if $ar\varrho > dr + d\varrho$, then 'tis $-f$.

In *Converging Rays*, falling on a double *Convex*, or *Concave*, (where 'tis $-d$, and $\pm r$, $\pm\varrho$) $- adr\varrho \div \overline{\mp dr \mp d\varrho - ar\varrho} = \pm f$; But in the *Concave*, if $ar\varrho > dr + d\varrho$, then 'tis $+f$.

In *Diverging*, or *Converging Rays*, falling on a *Meniscus*, $\mp adr\varrho \div \overline{\mp dr \pm d\varrho + ar\varrho} = + f$, whilst $dr - d\varrho >$ or $< ar\varrho$; But if $dr - d\varrho <$, or $> ar\varrho$, 'tis $= - f$.

In *Parallel Rays*, falling on a *Double Convex*, (where $d = \infty$) $f = a\varrho \div \overline{r + \varrho}$ or (in *Glass*, where $m : n :: 3 : 2$) $= 2r\varrho \div \overline{r + \varrho}$, Th. if $r = \varrho$, then $f = r$.

In *Diverging Rays*, on a *Plano-Convex*, (where $r = \infty$) $f = ad\varrho \div \overline{d - a\varrho}$.

If, $t = 2r$, & $r = \varrho$, and $d = \infty$, then $f = \overline{2nr - mr} \div \overline{2m - 2n}$, = (in a Sphere of *Glass*) $\frac{3}{2} r$, or in a Sphere of *Water*, (where $m : n :: 4 : 3$) $f = r$.

And

And wherever the Rays, which come from all the Points of any Object, meet again in so many Points, after they have been made to Converge by Reflexion or Refraction, there will they make a Picture of the Object. *Now, a Lens being given, to find the Distance, whereat an Object being plac'd, shall be Represented as large as the Object it self*; Here $d = f$, Th. $2 a r \rho \div \overline{r + \rho} = d$; or if $r = \rho$, then $d = ar$; or, in *Glass* $d = 2r$: In a *Plano Convex*, $d = 2ar$, i. e. in *Glass*, $d = 4r$, This is of singular use for Drawing things in their just Magnitude, by Transmitting the Species by a Glass into a dark Room, whereby, not only the True Figure and Shades are perfectly given, but also the Colors nearly as Vivid as the Life. 'Tis also easie from hence, to *Magnify* or *Diminish* an Object in any given Proportion. An Object seen by *Reflexion* or *Refraction*, appears in that Place from whence the Rays, after their last *Reflexion* or *Refraction*, diverge in falling on the Spectator's Eye. Whence, *As the distance of the Object from the Lens, is to the Diameter of the Object's Magnitude, so is the Distance of the Image to its Diameter.* And if the Object be seen, thro' two or more Glasses, every Glass shall make a New Image, and the Object shall appear in the Place, and of the Bigness of the last Image: Upon this depends the Theory of *Microscopes* and *Telescopes*.

Having given (r) the Rad. of one Segment of a *Lens*; to find the Rad. of the *Convexity* or *Concavity*, necessary to make a vastly distant Object be represented at a given *Focus*. Since r, a, f, and d, are given, Th. $drf \div \overline{adr - df + arf} =$ (because $d = \infty$) $rf \div \overline{ar - f} = +\rho$, or (if $f > ar$) $= -\rho$, and $rf \div \overline{f - ar} =$ Radius of the Concavity.

NOTE, That by putting $2rx - xx =$ to the Value of y^2 in any Curve, and rejecting the Terms concern'd with the Powers of x, the Radius of a Circle Equicurve with that of the *Lens*, at the Vertex, is determin'd, (and in the *Conic Sections* will be $= \frac{1}{2}$ the *Parameter*); this substituted in the room of r, ρ, will give the *Theorem* for a *Lens* of any given Figure. And where the Rays are Reflected from (GL) the Concave Surface of a *Speculum*, (bec. 'tis $+d$, $+m$, $-f$, $-r$, $-n$, and $m = n$,) $f = dr \div \overline{2d - r}$.

FINIS.