GEORG CANTOR AND THE BATTLE FOR TRANSFINITE SET THEORY

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Veniet tempus, quo ista quae nunc latent, in
lucem dies extrahat et longioris aevi diligentia.
--I Corinthians

The substance of Georg Cantor's life's work is well-known: in developing what he called the arithmetic of transfinite numbers he gave mathematical content to the idea of the actual infinite.[1] In so doing he laid the groundwork for abstract set theory and made significant contributions to the foundations of the calculus and to the analysis of the continuum of real numbers. Cantor's most remarkable achievement was to show, in a mathematically rigorous way, that the concept of infinity is not an undifferentiated one. Not all infinite sets are the same size, and consequently, infinite sets can be compared with one another.

So shocking and counter-intuitive were Cantor's ideas at first that the eminent French mathematician Henri Poincare condemned the theory of transfinite numbers as a “disease” from which he was certain mathematics would someday be cured. Leopold Kronecker, one of Cantor's teachers and among the most prominent members of the German mathematics establishment, even attacked Cantor personally, calling him a “scientific charlatan,” a “renegade” and a “corrupter of youth.”[2] It is also well-known that Cantor suffered throughout his life from a series of “nervous breakdowns” which became increasingly frequent and debilitating as he got older. The breakdowns were probably the symptoms of an organic mental illness - in fact, this has only recently been confirmed by an account given by one of the physicians who actually attended Cantor when he was a patient at the Halle Nervenklinik during the last years of Cantor's life, from 1913-1918. In the words of the psychiatrist Karl Pollitt:

> As a young assistant I treated a prominent professor of mathematics [Georg Cantor] who had to be admitted to the clinic because of the recurrence of a cyclic manic-depression.[3]

Nevertheless, it was all too easy for his early biographers to present Cantor, who was trying to defend his complex theory but was suffering increasingly long periods of mental breakdown, as the hapless victim of the persecutions of his contemporaries. But the more sensational accounts of Cantor's life distort the truth by trivializing the genuine intellectual concerns that motivated some of the most thoughtful contemporary opposition to Cantor's theory. They also fail to credit the power and scope of the defense Cantor offered for his ideas in the battle to win acceptance for transfinite set theory. At first he too resisted the implications of his research because he had always believed the idea of the actual infinite could not be consistently formulated and so had no place in rigorous mathematics.

Nevertheless, by his own account he soon overcame his "prejudice" regarding the transfinite numbers because he found they were indispensable for the further development of his mathematics. Because of his own early doubts he was able to anticipate opposition from diverse quarters, which he attempted to meet with philosophical and theological arguments as well as mathematical ones. Moreover, when he was called on to respond to his critics, he was able to muster his ideas with considerable force. His mental illness, as I will argue in a moment, far from playing an entirely negative role, may well have contributed in its manic phases to the energy and single-mindedness with which he promoted and defended his theory. Just as the theological dimension of Cantor's understanding of the infinite also reassured him -- in fact convinced him -- of its absolute truth, regardless of what opponents like Kronecker might say against the theory.[5]

But before it is possible to appreciate the origins, scope and significance of Cantor's battle to win acceptance for transfinite set theory, it will be helpful to say something, briefly, about his life and the early development of set theory.

GEORG CANTOR (1845 -1918)

Georg Ferdinand Ludwig Philip Cantor was born on March 3, 1845, in St Petersburg. His mother, a Roman Catholic, came from a family of notable musicians; his father, the son of a Jewish businessman, was also a successful tradesman, but a devout Lutheran, having been raised in a Lutheran mission in St Petersburg. Cantor's father, it should be added, passed his own deep religious convictions on to his son. According to Eric Temple Bell's widely-read book Men of Mathematics, first published in 1937, Georg Cantor's insecurities in later life stemmed from a ruinous Freudian conflict with his father, but surviving letters and other evidence concerning their relationship indicate quite the contrary. Georg's father appears to have been a sensitive man who was attentive to his children and took a special but not coercive interest in their welfare and education of his eldest son.[6]

When the young Cantor was still a child the family moved from Russia to Germany, and it was there he began to study mathematics. After receiving his doctorate from the University of Berlin in 1868 for a dissertation on the theory of numbers; two years later he accepted a position as privatdozent, or instructor, at the University of Halle, a respected institution but not as prestigious for mathematics as the universities at Gottingen or Berlin. One of his colleagues at Halle was Heinrich Eduard Heine. Heine was then working on the theory of trigonometric series, and he encouraged Cantor to take up the difficult problem of the uniqueness of such series. In 1872, when he was twenty-seven, Cantor published a paper that included a very general solution to the problem, along with his theory of real numbers and the seeds of what later would lead to his theory of transfinite sets and numbers. Cantor's first paper on trigonometric series actually appeared two years earlier in 1870, and showed that if a function is continuous throughout an interval, then its representation by a trigonometric series is unique. His next step was to relax the requirement that the function be continuous throughout the interval by allowing any finite number of exceptional points. In 1872 Cantor's search for an even more general statement of his uniqueness theorem led to a proof that so long as the exceptional points are distributed in a carefully specified way, there can even be infinitely many of them.

The most important step in the proof of the new result was the precise description of these infinite sets of exceptional points -- which Cantor called sets of the first species -- namely sets P whose derived sets P^v of limit points for some finite value of v is eventually empty. In order to describe with precision the actual structure of his point sets of the first species, Cantor found that he needed a rigorous theory of real numbers to make possible rigorous analysis of the continuum.

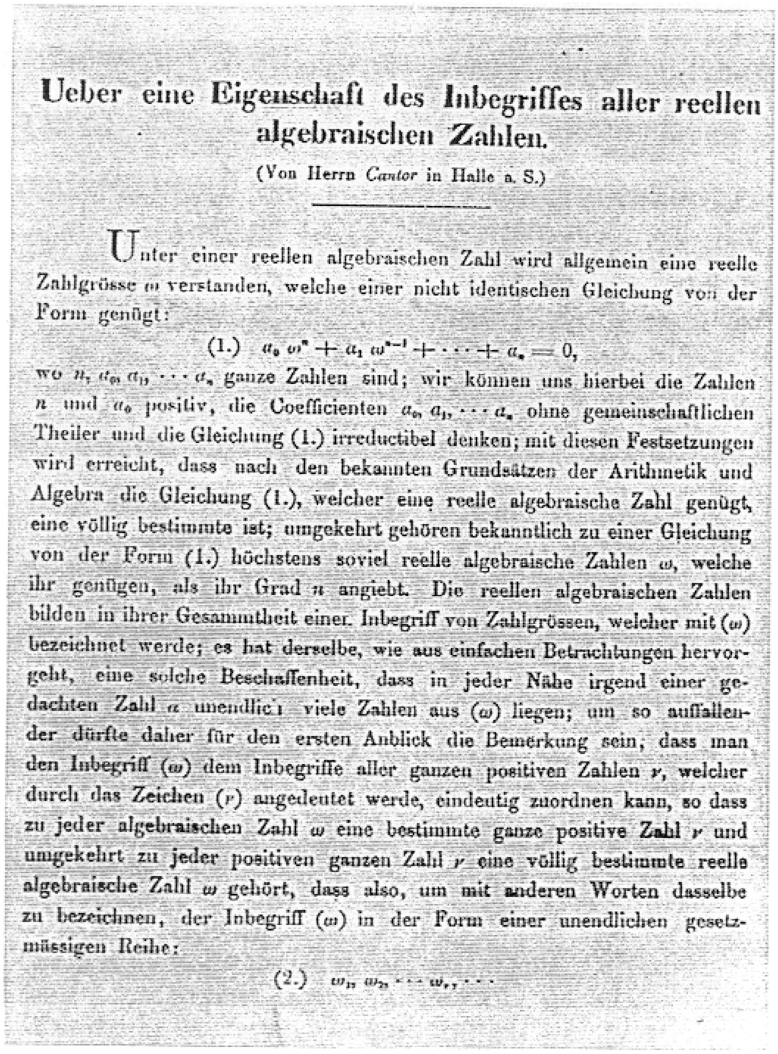
CANTOR AND DEDEKIND: REAL NUMBERS

Cantor was not alone in studying the properties of the continuum in rigorous detail. In 1872, the same year Cantor's paper appeared, the German mathematician Richard Dedekind also published an analysis of the continuum that was based on infinite sets. In his paper Dedekind articulated an idea that Cantor later made more precise:

> The line is infinitely richer in point-individuals than is the domain of rational numbers in number-individuals.[8]

Dedekind's statement, however, conceals a serious weakness. If anyone had asked Dedekind how much richer the infinite set of points in the continuum was than the infinite set of rational numbers, he could not have replied. Cantor's major contribution to this question was published in 1874 in Crelle's Journal.

What Cantor showed was non-denumerability of the real numbers, a discovery that was soon to transform much of modern mathematics. Cantor's paper was short, three pages, and bore a very strange title: "On a Property of the Collection of All Real Algebraic Numbers."[9]



Ueber eine Eigenschaft des Inbegriffes aller reellen algebraischen Zahlen.

(Von Herrn *Cantor* in Halle a. S.)

Unter einer reellen algebraischen Zahl wird allgemein eine reelle Zahlgrösse ω verstanden, welche einer nicht identischen Gleichung von der Form genügt:

$$(1.)\quad a_0\,\omega^n + a_1\,\omega^{n-1} + \cdots + a_n = 0,$$

wo $n, a_0, a_1, \cdots a_n$ ganze Zahlen sind; wir können uns hierbei die Zahlen n und a_0 positiv, die Coefficienten $a_0, a_1, \cdots a_n$ ohne gemeinschaftlichen Theiler und die Gleichung (1.) irreductibel denken; mit diesen Festsetzungen wird erreicht, dass nach den bekannten Grundsätzen der Arithmetik und Algebra die Gleichung (1.), welcher eine reelle algebraische Zahl genügt, eine völlig bestimmte ist; umgekehrt gehören bekanntlich zu einer Gleichung von der Form (1.) höchstens soviel reelle algebraische Zahlen ω, welche ihr genügen, als ihr Grad n angiebt. Die reellen algebraischen Zahlen bilden in ihrer Gesammtheit einen Inbegriff von Zahlgrössen, welcher mit (ω) bezeichnet werde; es hat derselbe, wie aus einfachen Betrachtungen hervorgeht, eine solche Beschaffenheit, dass in jeder Nähe irgend einer gedachten Zahl α unendlich viele Zahlen aus (ω) liegen; um so auffallender dürfte daher für den ersten Anblick die Bemerkung sein, dass man den Inbegriff (ω) dem Inbegriffe aller ganzen positiven Zahlen ν, welcher durch das Zeichen (ν) angedeutet werde, eindeutig zuordnen kann, so dass zu jeder algebraischen Zahl ω eine bestimmte ganze positive Zahl ν und umgekehrt zu jeder positiven ganzen Zahl ν eine völlig bestimmte reelle algebraische Zahl ω gehört, dass also, um mit anderen Worten dasselbe zu bezeichnen, der Inbegriff (ω) in der Form einer unendlichen gesetzmässigen Reihe:

$$(2.)\quad \omega_1, \omega_2, \cdots \omega_\nu, \cdots$$

No one scanning the title of this short paper, however, would have guessed that this was the paper that disclosed Cantor's revolutionary discovery of the non-denumerability of the continuum of real numbers. Instead, the article bore a deliberately misleading title suggesting that its major result was a theorem about algebraic numbers, thus failing even to hint at the more significant point the paper actually contained. What could possibly have prompted Cantor to choose such an inappropriate title for what now, in retrospect, strikes any modem reader as one of the most important discoveries in modem mathematics?

At the time Cantor originally wrote the paper late in 1873, he was nearly thirty, but only at the beginning of his mathematical career. Prior to joining the faculty of Halle University in 1869, he had studied with the greatest mathematicians of his day in Berlin -- Kummer, Weierstrass and Kronecker. In Weierstrass's seminar Cantor had already used the method of one-to-one correspondences to show the equivalence of the set of rational numbers Q and the natural numbers N, establishing that Q is denumerable even though it is dense and the set of integers is not. This was the same device Cantor later used in the paper of 1874 to show that the set of algebraic numbers is denumerable.

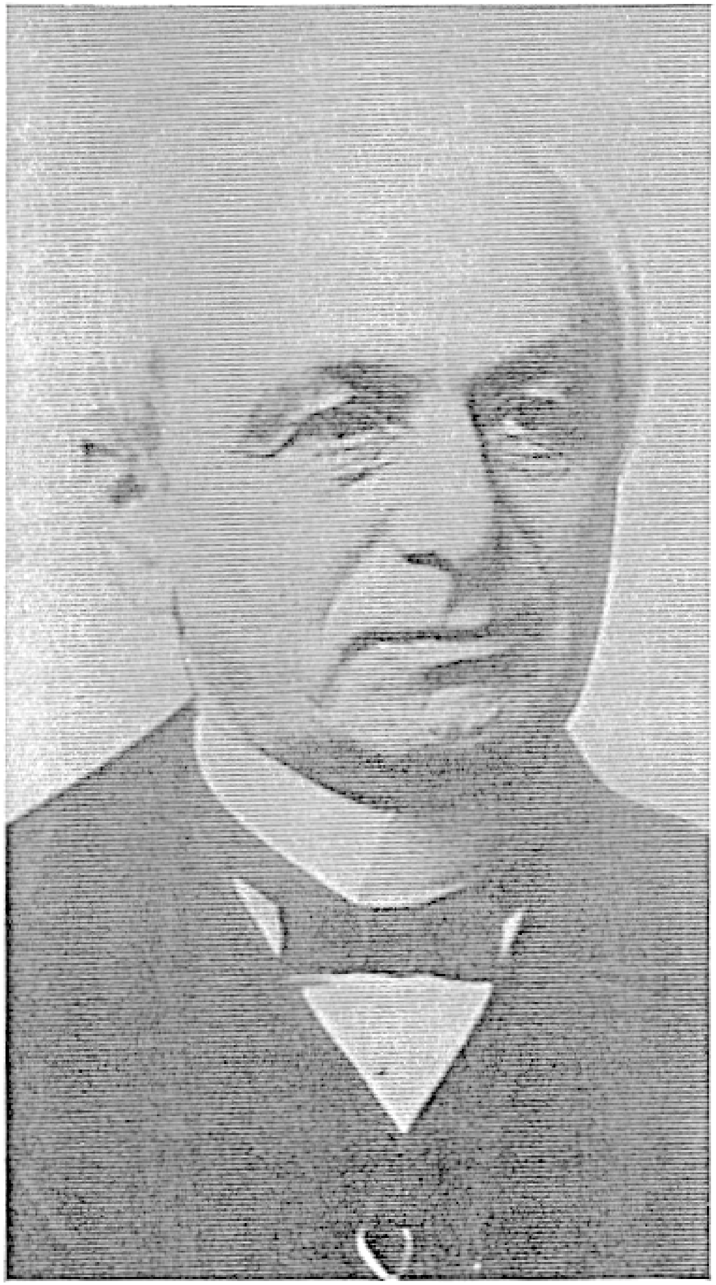
By 1872, Cantor was increasingly concerned with the structure of the continuum, largely because of the interesting questions raised by his uniqueness theorems concerning trigonometric series representations. These depended upon his introduction of point sets of the first species, which were always denumerable sets. Given a point set P, Cantor defined the set of all its limit points as P'. The set of limit points of P' constituted the second derived set P^2, etc. A set was said to be of the first species so long as P^n was empty for some finite value of n. If he could show that the continuum was also denumerable (as the rational and algebraic numbers had proven to be), then he could also hope to characterize the continuum in terms of sets of the first species.

Although Cantor admitted in a letter to Dedekind (December 2, 1873) that it might seem more reasonable to assume that no such one-to-one correlation was possible between the real numbers and the natural numbers he could find no reason to make this assumption. On the other hand, he knew that if he could show that such a mapping were impossible, this would provide a new proof of Liouville's theorem for the existence of transcendental numbers.[10]

Within a few days Cantor had finally found the answer, and wrote to Dedekind on December 7, 1873, explaining that it was not possible to provide any one-to-one correspondence between N and R. Cantor had a chance to show his proof to Weierstrass at the end of the month when he was in Berlin. Weierstrass was impressed, and urged Cantor to publish the result. Within weeks it appeared in <u>Crelle's Journal.</u> But as already noted, it bore the clearly inappropriate title: "On a Property of the Set of All Real Algebraic Numbers."

Back now to the question of why Cantor should consciously have chosen a deliberately inaccurate title. Walter Purkert and Hans Joachim Ilgauds have recently suggested one answer, in their book <u>Georg Cantor.</u> They argue that Weierstrass was primarily interested in Cantor's result about the denumerability of the algebraic numbers; therefore Cantor stressed this result in the title of his paper.[11] Although Cantor told Dedekind of Weierstrass's interest in the denumerability of the algebraic numbers (a result Weierstrass soon applied to produce an example of a continuous, non-differentiable function), the real reason why he had given such a "restrained character" to his exposition was due in part, Cantor said, to "conditions in Berlin."[12] Whatever Cantor may have meant by this cryptic remark (which he never explained further to Dedekind), what could it have had to do with the title of his paper?

The answer hinges on another of Cantor's teachers in Berlin, Leopold Kronecker. Having studied with Kronecker, Cantor was well acquainted with his work in number theory and algebra, and with his highly conservative philosophical views with respect to mathematics. By the early 1870's, Kronecker was already vocal in his opposition to the Bolzano-Weierstrass theorem, upper and lower limits, and to irrational numbers in general. Kronecker's later pronouncements against analysis and set theory, as well as his adamant and well-known insistence upon using the integers to provide the only satisfactory foundation for mathematics, were simply extensions; of these early views.[13]



With this in mind, it is not unreasonable to suspect: that Cantor had good reason to anticipate Kronecker's opposition to his proof of the non-denumerability of the real numbers. Certainly any result (such as Cantor's) that confirmed the existence of transcendental numbers would have been criticized by Kronecker. Worse yet, Kronecker was on the editorial board of the journal to which Cantor submitted his paper. Had Cantor been more direct with a title like "The set of real numbers is non-denumerably infinite," or "A new and independent proof of the existence of transcendental numbers," he could have counted on a strongly negative reaction from Kronecker. After all, when Lindemann later established the transcendence of π in 1882, Kronecker asked what value the result could possibly have, since irrational numbers did not exist. [14]

As Cantor contemplated publishing his paper in 1874, an innocuous title was clearly a strategic choice. Reference only to algebraic numbers would have had a much better chance of passing Kronecker's eye unnoticed, for there was nothing to excite either immediate interest or censure.

If the idea that Cantor may have harbored fears about Kronecker's opposition to his work at such an early date seems unwarranted, it is worth noting that Kronecker had already tried to dissuade Cantor's colleague at Halle, H.E. Heine, from publishing an article on trigonometric series in Crelle's Journal. Although Heine's article eventually appeared, Kronecker was at least successful in delaying its appearance (about which Heine was particularly vocal in letters to Schwarz, also a friend of Cantor.[15] Here the underlined portion of Heine's letter to Schwarz of May 26, 1870, explains Kronecker's efforts to prevent publication of Heine's paper. Doubtless Schwarz and Heine would both have brought Kronecker's readiness (and ability) to block ideas with which he disagreed to Cantor's attention). Indeed, several years later Kronecker also delayed publication of Cantor's 1878 paper on the invariance of dimension. This so angered Cantor that he never submitted anything to Crelle's Journal again. A decade later he regarded Kronecker as both a private and a public menace -- not only because he was condemning set theory openly, but Weierstrassian analysis too.[16]

There was a positive side, however, to Kronecker's opposition to Cantor's work, for it forced him to evaluate the foundations of set theory as he was in the process of creating it. This concern prompted long philosophical passages in Cantor's major publication of the 1880's on set theory, his Grundlagen einer allgemeinen Mannigfaltigkeitslehre of 1883. It was here that Cantor issued one of his most famous pronouncements about mathematics, namely that the essence of mathematics is exactly its freedom.[17] This was not simply an academic or philosophical message to his colleagues, for it carried as well a hidden and deeply personal subtext. It was, as he later admitted to David Hilbert, a plea for objectivity and openness among mathematicians. This, he said, was directly inspired by the oppression and authoritarian closed-mindedness that he felt Kronecker represented, and worse, had wielded in a flagrant and damaging way against those he opposed.

Thus at the very beginning of his career, and even before he had begun to develop any of his more provocative ideas about transfinite set theory, Cantor had experienced his first bitter tastes of Kronecker's opposition to his work. Doubtless Cantor knew that he could expect more trouble in the future.

It was also in 1883 that Cantor first tried vigorously to establish the Continuum Hypothesis in the version that the set of real numbers was the next largest after the denumerable set of natural numbers. Cantor's unsuccessful efforts, however, to prove the Continuum Hypothesis caused him considerable stress and anxiety. Early in 1884 he thought he had found a proof, but a few days later he reversed himself completely and thought he could actually disprove the hypothesis. Finally he realized that he had made no progress at all, as he reported in letters to Mittag-Leffler. All the while he had to endure mounting opposition and threats from Kronecker, who said he was preparing an article that would show that “the results of modern function theory and set theory are of no real significance."[18]

Soon thereafter, in May of 1884, Cantor suffered the first of his serious nervous breakdowns. Although frustration over his lack of progress on the Continuum Hypothesis or stress from Kronecker's attack may have helped to trigger the breakdown, it now seems clear that such events had little to do with its underlying cause. The illness took over with startling speed and lasted for somewhat more than a month. At the time, only the hyperactive phase of manic-depression was recognized as a symptom; when Cantor “recovered” at the end of June 1884, and entered the depressive phase, he complained that he lacked the energy and interest to return to rigorous mathematical thinking. He was content to take care of trifling administrative matters at the university, but felt capable of little more.

Although Cantor eventually returned to mathematics, he also became increasingly absorbed in other interests. He undertook a study of English history and literature and became engrossed in a scholarly diversion that was taken with remarkable seriousness by many people at the time, namely the supposition that Francis Bacon was the true author of Shakespeare's plays. Cantor also tried his hand without success at teaching philosophy instead of mathematics, and he began to correspond with several theologians who had taken an interest in the philosophical implications of his theories about the infinite. This correspondence was of special significance to Cantor because he was convinced that the transfinite numbers had come to him as a message from God. But more of the significance of this, as already promised, in a moment.

There is still one last element of Cantor's technical development of transfinite set theory that needs to be mentioned as part of his continuous efforts to mount a convincing and satisfactory mathematical defense of his ideas -- namely the nature and status of the transfinite cardinal numbers.

The evolution of Cantor's thinking about the transfinite cardinals is curious, because, although the alephs are probably the best-known legacy of Cantor's creation, they were the last part of his theory to be given either rigorous definition or a special symbol. Indeed, it is difficult in the clarity of hindsight to reconstruct the obscurity within which Cantor must have been groping, and his work up to now has been discussed here largely as if he had already recognized that the power of an infinite set could be understood as a cardinal number. In fact, beginning in the early 1880's, Cantor first introduced notation for his infinite (actually transfinite) sequence of derived sets P^{v}, extending them well beyond the limitation he had earlier set himself to sets of the first species. At this time he only spoke of the indexes as “infinite symbols” with no reference to them in any way as numbers.[20]

By the time he wrote the Grundlagen in 1883, the transfinite ordinal numbers had finally achieved independent status as numbers, and were given the familiar omega notation. However, there was no mention whatsoever of transfinite cardinal numbers, although Cantor clearly understood that it is the power of a set that establishes its equivalence (or lack thereof) with any other set. Nevertheless, he carefully avoided any suggestion that the power of an infinite set could be interpreted as a number.

Soon, however, he began to equate the two concepts, and by September of 1883 did so in a lecture to mathematicians at a meeting in Freiburg.[21] Even so, no symbol was yet provided for distinguishing one transfinite cardinal number from another. Since he had already adopted the symbol □ to designate the least transfinite ordinal number. When Cantor finally introduced a symbol for the first transfinite cardinal number, it was borrowed from the symbols already in service for the transfinite ordinals. By 1886, in correspondence, Cantor had begun to represent the first transfinite cardinal as □; the next largest he denoted □. This notation was not very flexible, and within months Cantor realized he needed a more general notation capable of representing the entire ascending hierarchy of transfinite cardinals. Temporarily, he used fraktur “o”s -- obviously a derivative from his omegas, to represent his sequence of cardinal numbers.

For a time Cantor actually used superscripted stars, bars and his fractur “o"s interchangeably for transfinite cardinals without feeling any need to decide upon one or the other as preferable. But in 1893 the Italian mathematician Giulio Vivanti was preparing a general account of set theory, and Cantor realized it was time to adopt a standard notation. Only then did he choose the alephs for the transfinite cardinals because he thought the familiar Greek and Roman alphabets were already too widely employed in mathematics for other purposes. His new numbers deserved something distinct and unique. Thus he chose the Hebrew letter aleph -- which was readily available in the type fonts of German printers. The choice was particularly clever, as Cantor was pleased to admit, because the Hebrew aleph was also a symbol for the number one. Since the transfinite cardinal numbers were themselves infinite unities, the aleph could be taken to represent a new beginning for mathematics. Cantor designated the cardinal number of the first number class aleph_1 in 1893, but by 1895 changed his mind to aleph_0, the number he had previously called □, the cardinal number of the second number class was designated aleph_1.[22]

Cantor made his last major contributions to set theory in 1895 and 1897. He had already used his famous method of diagonalization in 1891 to show that the cardinal number of any set is always smaller than the cardinal number of the set of all its subsets.[23] A few years later he presented a corollary to this result, that the cardinal number of the continuum is equal to 2^{aleph0}, and he hoped this result would soon lead to a solution of the continuum hypothesis -- which could now be expressed simply as $2^{\text{aleph0}} = \text{aleph}_1$.

The arguments in Cantor's proof about the cardinal number of the power set of subsets, however, led to far different conclusions. The most important of these was reached by Bertrand Russell in 1903. Russell showed that a paradox can be derived in set theory by considering all sets that do not include themselves as members. Russell's paradox suggested there was something fundamentally wrong with Cantor's definition of a set, and the consequences of this realization immediately became an important problem m 20th-century mathematics.[24]

Even before Bertrand Russell, however, Cantor had already come upon his own version of the paradoxes of set theory in the form of contradictions he associated with the idea of a largest ordinal or cardinal number. This was all explained in letters first to Hilbert in 1897, and then to Dedekind in 1899. In the draft of his letter to Dedekind of August 3, 1899, for example, amid the many additions and crossed-out material, next to the A in the left-hand margin, Cantor wrote “The system □ of all numbers is an inconsistent ‘absolut unendliche Vielheit’” -- an inconsistent absolutely infinite aggregate.[25]

But it is possible, as I want to suggest now in conclusion, that Cantor may well have been aware of the paradoxes of set theory much earlier, perhaps as early as the 1880's, when his difficulties with Kronecker were weighing on his mind and he was just beginning to experience his first serious technical problems with set theory.

For example, as early as his Grundlagen of 1883, Cantor referred to collections that were too large, he said, to be comprehended as well defined, completed, unified entities. Unfortunately, he wrote obscurely, with references to absolute sets in explicitly theological terms, explaining that “the true infinite or Absolute, which is in God, permits no determination.”[26]

In a footnote accompanying the Grundlagen, he went further and explained that "the absolute infinite succession of numbers [Zahlenfolge] seems to me therefore to be an appropriate symbol of the absolute."[27] Was this a hint that he already understood that the collection of all transfinite ordinal numbers was inconsistent, and therefore not to be regarded as a set? Later, Cantor said that was exactly his meaning – a veiled sign, even then, that he was aware of the paradoxical results that followed from trying to determine what transfinite ordinal number should correspond to the well-ordered set of all transfinite ordinal numbers.

By the mid-1890s Cantor could no longer be so vague about absolute entities, and was forced to be much more explicit about the paradoxes resulting from consideration of the sets of all transfinite ordinal or cardinal numbers. The solution Cantor then devised for dealing with such mathematical paradoxes was simply to bar them from set theory. Anything that was too large to be comprehended as a well defined, unified, consistent set was declared inconsistent. These were "absolute" collections, and lay beyond the possibility of mathematical determination. This, in essence, is what Cantor communicated first to Hilbert in 1897, and somewhat later to Dedekind in his letters of 1899.[28]

Whatever the extent of Cantor's awareness of the paradoxes may have been in the early 1880's, he was certainly sensitive to Kronecker's growing and increasingly vocal opposition. Above all, it is clear that explicitly philosophical concerns expressed in his Grundlagen were strategically crucial in Cantor's opinion for a comprehensive defense of his new theory. Not only was this unusual at the time, but it still is. When Mittag-Leffler asked Cantor's permission to publish French translations of Cantor's papers on set theory for his newly founded journal Acta Mathematica, he persuaded Cantor to omit all of the philosophical portions of the Grundlagen as unnecessary (and possibly repugnant in Mittag-Leffler's view) to mathematicians who might find the theory of interest but the philosophy unacceptable.[29]

The philosophical arguments, however, were essential to Cantor, if not to Mittag-Leffler. They were essential because they were part of the elaborate defense he had begun to construct to subvert opposition from any quarter, but especially from Kronecker. The ploy was to advance a justification based upon the freedom of mathematics to admit any self-consistent theory. Applications might eventually determine which mathematical theories were useful, but for mathematicians, Cantor insisted that the only real question was consistency. This of course was just the interpretation he needed to challenge an established mathematician like Kronecker. Cantor clearly felt obliged, early in his career, to plead as best he could for a fair hearing of his work. So long as it was self-consistent, it should be taken as mathematically legitimate, and the constructivist, finitist criticisms of Kronecker might be disregarded by most mathematicians for whom consistency alone should be the viable touchstone.

Cantor put his philosophy about the freedom of mathematics into action early in the 1890's, when his career had reached the point where he could do more than simply write about it. During the 1880's he had already begun to lay the strategic foundations for an independent union of mathematicians in Germany. The specific goal of such a union, as he often made clear in his correspondence, was to provide an open forum, especially for young mathematicians. The union (as Cantor envisaged it) would guarantee that anyone could expect free and open discussion of mathematical results without prejudicial censure from members of the older establishment whose conservatism might easily ruin the career of an aspiring mathematician. Above all, this was needed in cases where the ideas in question were at all new, revolutionary or controversial.

Cantor labored intensively for the creation of the Deutsche Mathematiker-Vereinigung. Eventually agreement was reached and the Union of German Mathematicians held its first meeting in conjunction with the annual meeting of the Gesellschaft Deutscher Naturforscher und Aerzte, which met at Halle in 1891. Cantor was elected the Union's first President, and at its inaugural meeting he presented his now famous proof that the real numbers are non-denumerable using his method of diagonalization.[30]

The German Union was not the end of Cantor's vision. He also recognized the need to promote international forums as well, and began lobbying shortly after formation of the DMV for regular international congresses. These were eventually organized through the cooperative efforts of many mathematicians, and not directly, it should be added, as a result of Cantor's exclusive efforts by any means. The first of these was held at Zurich in 1897, the second at Paris in 1900.

Promoting new avenues for discussion of mathematics was one way Cantor reacted to opposition of the sort his own research had provoked. Despite criticisms, especially from Kronecker, Cantor persevered, even in the face of his own repeated failure to resolve some of the most basic questions about set theory (notably his Continuum Hypothesis), and even though he began to suffer increasingly serious cycles of manic-depression. Ironically, like his conflicts with Kronecker, Cantor's manic-depression may also have served a useful purpose. In his own mind it was closely linked to the infallible support set theory drew from his strongly-held religious convictions. Letters (and the testimony of colleagues who knew him) reveal that Cantor believed he was chosen by God to bring the truths of set theory to a wider audience. He also regarded the successive waves of manic-depression that began to plague him in the 1880's -- peaks of intense activity followed by increasingly prolonged intervals of introspection -- as divinely inspired. Long periods of isolation in hospital provided opportunities for uninterrupted reflection during which Cantor envisioned visits from a muse whose voice reassured him of the absolute truth of set theory, whatever others might say about it.

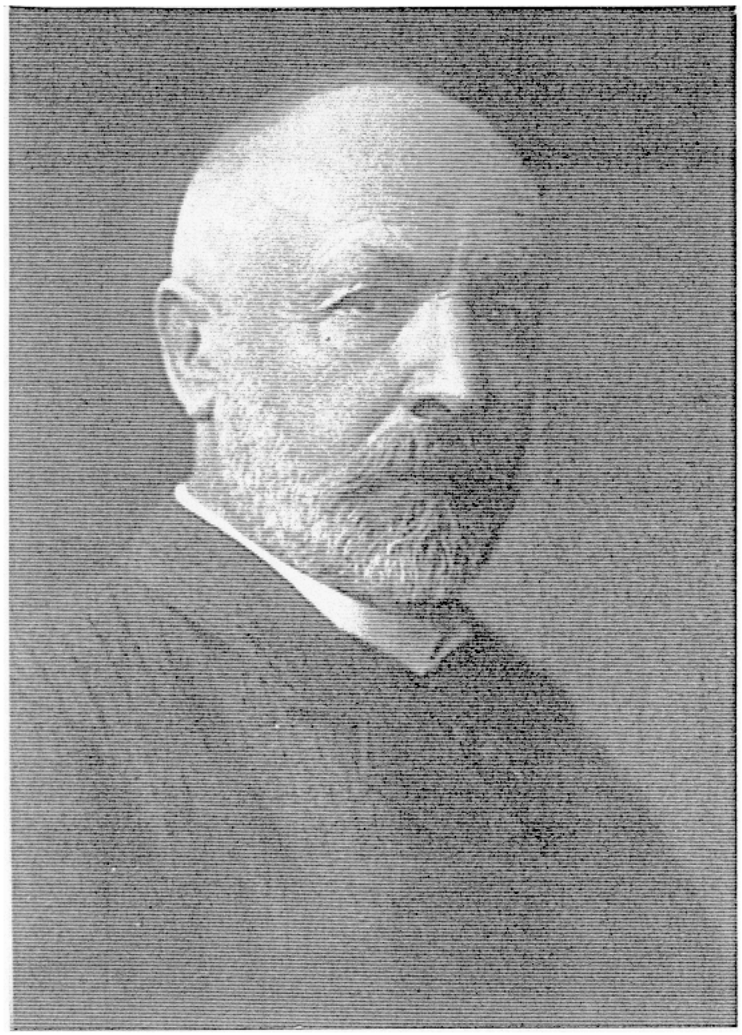
In promoting set theory among mathematicians, philosophers and theologians (he even wrote to Pope Leo XIII at one point on the subject of the infinite), Cantor was convinced he would succeed in securing the recognition that transfinite set theory deserved.[31] By stressing self-consistency and the intrinsic freedom of mathematics, he also advanced an essential element of any intellectual inquiry. The mind must be free to pursue the truth wherever it might lead. Inspiration should be encouraged, not confounded by arbitrary prejudice, and for Cantor this meant a tolerance for theories judged upon standards of consistency and utility.

THE IMAGES OF SET THEORY

In closing, there is yet another important connection between Cantor's mental illness and his mathematics worth considering. Certain documents suggest that in addition to enforcing periodic intervals of contemplation and withdraw from daily affairs, Cantor's periods of depression were not wholly unproductive, and in fact he was often able to pursue his mathematical ideas in the solitude of the hospital or quietly at home. The illness may, in fact, have supported his belief that the transfinite numbers had been communicated to him from God. In fact, as he noted in the third motto to his last publication, the Beitrage of 1895:

> The time will come when these things which are now hidden from you will be brought into the light.[32]

This is a familiar passage from the Bible, first Corinthians, and reflects Cantor's belief that he was an intermediary serving as the means of revelation. It also may have served to reflect Cantor's faith that despite any prevailing resistance to his work, it would one day enjoy recognition and praise from mathematicians everywhere.



It is easy, of course, to misinterpret the religious element in Cantor's thinking, as popularizers often do. This was certainly the case in an article that appeared not long ago in the French magazine La Recherche, which supplied the following caricatures to illustrate an expository article about Cantor, his religious convictions, psychological illness and transfinite set theory.[33] The first drawing depicts Cantor in ecstasy, as it were, receiving the divine message. In the second illustration, the figure with the gun of course is meant to be Kronecker -- with God helping Cantor to maintain his balance -- all of which rests precariously on a transfinite aleph.

But there is a very serious side to all of this, and it deserves to be emphasized. For example, following a long period of hospitalization in 1908, Cantor wrote to a friend in Gottingen, the British mathematician Grace Chisholm Young. As he described it, his manic depression took on a strikingly creative quality:

> A peculiar fate, which thank goodness has in no way broken me, but in fact has made me stronger inwardly, happier and more expectantly joyful than I have been in a couple of years, has kept me far from home -- I can also say far from the

> world... In my lengthy isolation neither mathematics nor in particular the theory of transfinite numbers has slept or lain fallow in me; the first publication in years which I shall have to make in this area is designated for the "Proceedings of the London Mathematical Society."[34]

Elsewhere, Cantor actually described his conviction about the truth of his theory explicitly in quasi-religious terms:

> My theory stands as firm as a rock; every arrow directed against it will return quickly to its archer. How do I know this? Because I have studied it from all sides for many years; because I have examined it from all sides for many years; because I have examined all objections that have ever been made against the infinite numbers, and above all because I have followed its roots, so to speak, to the first infallible cause of all created things.[35]

Later generations might dismiss the philosophy, look askance at his abundant references to St. Thomas or to the Church Fathers, overlook the metaphysical pronouncements and miss entirely the deeply religious roots of Cantor's later faith in the absolute truth of his theory.

But all these commitments contributed to Cantor's resolve not to abandon the transfinite numbers. Opposition seems to have strengthened his determination. His forbearance, as much as anything else he might have contributed, ensured that set theory would survive the early years of doubt and denunciation to flourish eventually as a vigorous, revolutionary force in 20th-century mathematics.

ENDNOTES

1. Much of the information presented here may be found in Joseph W. Dauben, Georg Cantor. His Mathematics and Philosophy of the Infinite. Cambridge, Mass.: Harvard University Press, 1979; reprinted Princeton: Princeton University Press, 1990. Additional details depend on archival material drawn from archives in Gottingen and Berlin, including the Deutsche Akademie der Wissenschaften (Berlin), the Deutsches Zentralarchiv in Merseburg, and the important holdings at the Institute Mittag-Leffler in Djursholm, Sweden.

2. Henri Poincare, "L' Avenir des mathematiques," in Atti del IV Congresso Internazionale dei Matematici. Rome. 1908. Rome: Accademia dei Lincei, 1909, pp. 167-182, esp. p. 182; Leopold Kronecker as quoted in A. Schoenflies, "Die Krisis in Cantor's mathematischem Schaffen," Acta Mathematica. 50 (1927), pp. 1-23, esp. p. 2.

3. K. Ponitz, "Shakespeare und die Psychiatrie," Therapie der Gegenwart. 12 (1964), pp. 1463-1478, esp. p. 1464; also quoted in Walter Purkert and Hans Joachim Dgauds, Georg Cantor. 1845-1918. Basel: Birkhauser Verlag, 1987, p, 79.

4. Among these, see Schoenflies 1927, although the most exaggerated though possibly best-known is that by E.T. Bell, Men of Mathematics. New York: Simon and Schuster, 1937, Chapter 29, pp. 555-579. For more balanced accounts, see Ivor Grattan-Guinness, “Towards a Biography of Georg Cantor,” Annals of Science, 27 (1971), pp. 345-391; and “Cantors Krankheit” in Purkert and Dgauds 1987, pp. 79-92.

5. See Dauben 1979/1990, pp. 284-291.

6. See the chapter, “Paradise Lost?” in Bell 1937, esp. pp. 560-561. Details of the close relationship between father and son are provided in H. Meschkowski, Probelme des Unendlichen: Werk un Leben Georg Cantors. Braunschweig: Vieweg, 1967, pp. 1-9. See also Dauben 1979/1990, pp. 272-280.

   A famous (and lengthy) letter Cantor's father sent his son on the occasion of his Confirmation, dated “Pfingsten, 1960,” is reproduced in A. Fraenkel, “Georg Cantor,” lahresbericht der Deutschen Mathematiker-Vereinigung. 39 (1930), pp. 189-266. The Whitsuntide letter appears on pp. 191-192. This biography also appears separately as Georg Cantor. Leipzig: B.. Teubner, 1930, and is reprinted in an abridged version in Georg Cantor, Gesammelte Abhandluni!en mathematischen und ohilosoohischen Inhalts. ed. E. Zermelo, Berlin: Springer, 1930, pp. 452-483; reprinted Hildesheim: Olms, 1966; Berlin: Springer, 1980, 1990. The Whitsuntide letter is given in part in H. Meschkowski 1967, p. 3, and in full in H. Meschkowski and W. Nilson, eds., Georg Cantor. Briefe. Berlin: Springer, 1990, pp. 18-21; an English translation appears in Dauben 1979/1990, pp. 274-276.

7. G. Cantor, “Uber die Ausdehnung eines Satzes aus der Theorie der trigonometrischen Reihen,” Mathematische Annalen. 5 (1872), pp. 123-132; in G. Cantor, Gesammelte Abhandlungen mathematischen und ohilosophischen Inhalts. ed. E. Zermelo, Berlin: J. Springer, 1932, pp. 92-102; reprinted Hildesheim: Olms, 1966; Berlin: Springer, 1980, 1990.

8. Richard Dedekind, Stetigkeit und irrationale Zahlen, Braunschweig: Vieweg, 1872, pp. 1~2; in R. Dedekind, Gesarnrnelte mathematische Werke, eds. R. Fricke, E. Noether and O. Ore, Braunschweig: Vieweg, 1930-1932.

9. G. Cantor, “On a Property of the Collection of All Real Algebraic Numbers,” Journal fur die reine und angewandte Mathematik, 77 (1874), pp. 258-262; in Cantor 1932, pp. 115-118.

10. Briefwechsel Cantor-Dedekind. eds. E. Noether and J. Cavailles, Paris: Hermann, p. 13.

11. Purkert and Ilgauds 1987, p. 46.

12. See Cantor's letter to Dedekind of December 25, 1873, in Briefwechsel Cantor-Dedekind, pp.16-17.

13. See Dauben 1979/1990, pp. 66-70.

14. Kronecker made this legendary remark during a lecture he gave at a meeting of the Berliner Naturforscher Versammlung in 1886. See H. Weber, "Leopold Kronecker," Mathematische Annalen, 43 (1893), pp. 1-25, esp. p. 15; "Leopold Kronecker," Jahresbericht der Deutschen Mathematiker Vereinigung. 2 (1893), pp. 5-23, esp. p. 19; A. Kneser, "Leopold Kronecker," Jahresbericht der Deutschen Mathematiker Vereinigung. 33 (1925), pp. 210-228, esp. p. 221; and J. Pierpont, "Mathematical Rigor, Past and Present," Bulletin of the American Mathematical Societv. 34 (1928), pp. 23-53, esp. p. 39.

15. Heine to Schwarz, May 26, 1870, in Dauben 1979/1990, pp. 66-67; the letter is transcribed in full, pp. 307-308.

16. See A. Fraenke11930, pp. 189-266, in Cantor 1932, pp. 458-460; Meschkowski 1967, p. 40; Dauben 1979/1990, pp. 69-70.

17. Cantor, Grundlagen einer allegemeinen Mannigfaltigkeitslehre. Ein mathematisch-ohilosophischer Versuch in der Lehre des Unendlichen. Leipzig: B. O. Teubner. In Cantor 1932, pp. 165-208, esp. p. 182.

18. Quoted in a letter from Cantor to Mittag-Leffler, January 26, 1884, in Schoenflies 1927, p. 5. For Cantor's letters to Mittag-Leffler in which he claimed first to have proved the Continuum Hypothesis, then to have disproved it, and finally that he was unable to establish either conclusion, see those he wrote between August 26 and November 14, 1884, all transcribed in Schoenflies 1927, pp. 16-18. See also Meschkowski 1991, p. 170.

19. See Cantor's letter to Mittag-Leffler dated June 21, 1884, in Schoenflies 1927, p. 9. Cantor's breakdown in the summer of 1884 is also discussed in Dauben 1979/1990, pp. 280-282. See as well Grattan-Guinness 1971, pp. 355-358 and 368-369.

20. Cantor, "Ueber unendliche, lineare Punktmannigfaltigkeiten," Part 4, Mathematische Annalen, **21** (1883), pp. 51-58; in Cantor (1932), pp. 157-164, esp. p. 160.

21. Cantor described his lecture at the Freiburg meeting of the section for mathematics of the Oesellschaft Deutscher Naturforscher und Aerzte in a letter to Wilhelm Wundt of October 5, 1883, transcribed in Meschkowski 1991, pp. 136-140.

22. Cantor explained his choice of the alephs to denote the transfinite cardinal numbers in a letter to Felix Klein of April 30, 1895. The original letter is in the Klein Nachlass, Universitatsbibliothek, Gottingen, and may also be read in a draft version in Cantor's letter-book for 1890-1895, pp. 142-143, also kept in the archives of the Niedersachsische Staats- und Universitatsbibliothek, Gottingen. See also Dauben 1979/1990, pp. 179-183; Meschkowski 1991, pp. 354-355.

23. G. Cantor, "Uber eine elementare Frage der Mannigfaltigkeits-lehre," Jahresbericht der Deutschen Mathematiker-Vereinigung,1 (1891), pp. 75-78; in Cantor (1932), pp. 278-280.

24. Russell explained the significance of his paradox in a famous letter to Gottlob Frege,

reprinted in J. van Heijenoort, From Frege to Godel. A Source Book in Mathematical Logic, Cambridge, Mass.: Harvard University Press, 1967, pp. 124-25; Frege's reaction is detailed in Appendix II to G. Frege, Grundgesetze der Arithmetik. begriffsschriftlich abgeleitet. II, Jena: Verlag Hermann Pohle; reprinted Hildesheim: Olms, 1962, pp. 127-143. For Russell's own views of the significance of the paradoxes of set theory, see B. Russell, Principles of Mathematics. Cambridge, England: Cambridge University Press, 1903; and "On Some Difficulties in the Theory of Transfinite Numbers and Order Types," Proceedings of the London Mathematical Society. 4 (1907), pp. 29-53.

25. The letter is transcribed in Cantor 1932, p. 445; in Meschkowski 1991, p. 408.

28. Cantor's letters to Hilbert have recently been published in Purkert and llgauds 1987, pp. 224-227; the letters to Dedekind were first published by Zermelo in Cantor 1932, pp. 443-450. For discussion of the Cantor-Dedekind correspondence, including problems with Zermelo's editing of the letters, see lvor Grattan-Guinness, "The Rediscovery of the Cantor-Dedekind Correspondence," Journal fur die reine und angewandte Mathematik. 76 (1974), pp. 104-139, and Dauben 1979/1990, pp. 351-352. See as well Meschkowski 1991, pp. 405-412.

29. See Mittag-Leffler's letter to Cantor of March 11, 1883. The letter is part of the Nachlass Mittag-Leffler kept at the lnstitut Mittag-Leffler in Djursholm, Sweden. See Dauben 1979/1990, p.96.

30. Dauben 1979/1990, p. 165.

31. Dauben 1979/1990, pp. 146-148.

32. See G. Cantor, "Beitrage zur Begrundung der transfiniten Mengen-lehre," Part I, Mathematische Annalen. 46 (1895), pp. 481-412, esp. p. 481; and Cantor 1932, p. 282.

33. Pierre Thuillier, "Dieu, Cantor et l'Infini," La Recherche, (December, 1977), pp. 1110-1116.

34. Cantor to Grace Chisholm Young, June 20, 1908, transcribed in H. Meschkowski, "Zwei unveroffentlichte Briefe Georg Cantors," Der Mathematikunterricht, 4 (1971), pp. 30-34; and Meschkowski 1991, pp. 453-454. For an English translation, see Dauben 1979/1990, p. 290.

35. Cantor to Heman, June 21, 1988, from a draft version in Cantor's letter-book for 1884-1888, pp. 179, in the archives of the Niedersachsische Staats- und Universitatsbibliothek, Gottingen. See also Dauben 1979/1990, p. 298.