

# RECIPROCALS AND RECIPROCAL ALGORITHMS IN MESOPOTAMIAN MATHEMATICS

DUNCAN J. MELVILLE

## 1. INTRODUCTION

Reciprocals play a significant role in Mesopotamian mathematics since division is performed as ‘multiplication by the reciprocal’. An important skill for a Mesopotamian scribe was the ability to find reciprocals, and there are a number of algorithms for achieving this. The product of a number and its reciprocal is 1. For any number  $n$ , we let  $\bar{n}$  denote the reciprocal. Then  $n\bar{n} = 1$ . In Mesopotamia, the notion of the ‘reciprocal’ only appears after the introduction of the abstract sexagesimal system, which utilizes a relative place value. No absolute scale of the numbers is indicated and so, in effect, we treat as a number and its reciprocal any pair of numbers whose product is a power of 60, and hence denoted by 1 in the sexagesimal system. For example, the reciprocal of 2 is 30 because  $2 \cdot 30 = 1$  in sexagesimal notation. Similarly, the reciprocal of 4 is 15. The closest analogy in modern life is with minutes and hours: half an hour is 30 minutes and a quarter of an hour is 15 minutes. It is often helpful to think of 30 as a half, 20 as a third, 15 as a quarter, and so on. In fact, there is evidence from early examples of reciprocal tables from the Ur III period that these pairs were thought of as factors of 60. It would be better to think of the pairs as multiplicative inverses rather than reciprocals, but the terminology is entrenched.

## 2. RECIPROCAL TABLES AND THE STANDARD TABLE

A sexagesimal number has a (finite) reciprocal if and only if its only prime divisors are 2, 3 and 5, the divisors of the base, 60. These are the so-called ‘regular’ numbers. This property holds for any base. For instance, in base 10, the prime divisors are 2 and 5, so 2 has a finite decimal reciprocal  $\frac{1}{2} = 0.5$ , and so does 5, where  $\frac{1}{5} = 0.2$ , but 3 does not. We cannot write  $\frac{1}{3}$  as a finite decimal. Since 60 has many more divisors than 10, there are many (small) regular numbers. For easy reference these were collected into a standard table, of which we have many surviving copies. The standard table contains reciprocal pairs for all regular numbers from 2 to 1, 21 = 81 = 9<sup>2</sup> (see Table 1)

Note that reciprocals come in pairs. The reciprocal of 2 is 30 and the reciprocal of 30 is 2. In terms of our notation, this means that  $\overline{(\bar{n})} = n$ . Hence, one can (and should) use the table in both directions, reading from left to right and from right to left.

The table is not difficult to construct starting with the basic pairs  $2 \sim 30$ ,  $3 \sim 20$ ,  $5 \sim 12$  and using a simple doubling and halving procedure, along with a couple of extra arguments. For example, from  $2 \sim 30$ , by doubling the first number and

| $n$ | $\bar{n}$ | $n$ | $\bar{n}$ | $n$  | $\bar{n}$ |
|-----|-----------|-----|-----------|------|-----------|
| 2   | 30        | 16  | 3,45      | 45   | 1,20      |
| 3   | 20        | 18  | 3,20      | 48   | 1,15      |
| 4   | 15        | 20  | 3         | 50   | 1,12      |
| 5   | 12        | 24  | 2,30      | 54   | 1,6,40    |
| 6   | 10        | 25  | 2,24      | 1    | 1         |
| 8   | 7,30      | 27  | 2,13,20   | 1,4  | 56,15     |
| 9   | 6,40      | 30  | 2         | 1,12 | 50        |
| 10  | 6         | 32  | 1,52,30   | 1,15 | 48        |
| 12  | 5         | 36  | 1,40      | 1,20 | 45        |
| 15  | 4         | 40  | 1,30      | 1,21 | 44,26,40  |

TABLE 1. The standard table of reciprocals

halving the other, we get  $4 \sim 15$ , then  $8 \sim 7,30$  (thinking of this as 7 and a half),  $16 \sim 3,45$ , giving  $32 \sim 1,52,30$  and  $1,4 \sim 56,15$ . Starting from  $3 \sim 20$ , the same process gives us the pairs  $6 \sim 10$ , then  $12 \sim 5$ , and  $24 \sim 2,30$  ending with  $48 \sim 1,15$ . Similarly, from the  $5 \sim 12$  pair, we get  $10 \sim 6$ , and  $20 \sim 3$ , then  $40 \sim 1,30$  and  $1,20 \sim 45$ . Since 9 is  $3^2$ , the reciprocal of 9 is the square of the reciprocal of 3, or  $20^2 = 6,40$ . Then  $18 \sim 3,20$ , so  $36 \sim 1,40$  and  $1,12 \sim 50$ . Similarly,  $25 = 5^2 \sim 12^2 = 2,24$ , and so  $50 \sim 1,12$ , which we already knew and could have used to find  $25 \sim 2,24$ . Finally, from  $9 \sim 6,40$ , we obtain  $27 \sim 2,13,20$ , then  $54 \sim 1,6,40$ , and  $1,21 \sim 44,26,40$ . The last flourish is to put  $1 \sim 1$ . The table is complete.

In Old Babylonian mathematics problems, most reciprocals that are needed are ones from the standard table. However, some times a student needed to find the reciprocal of a regular number not in the table, and for this situation, standard algorithms were developed.

### 3. A SIMPLE ALGORITHM FOR THE RECIPROCAL

A calculation using the following algorithm appears on a tablet from Ur, published as UET 6/2 295. The calculations were decoded by Friberg [1].

The algorithm works as follows. We are given a number  $n$  and want to find its reciprocal  $\bar{n}$ . The procedure is to view  $n$  as composed of two parts,  $n = a + b$ , where the tail,  $b$ , is a number in the standard table. Hence, its reciprocal,  $\bar{b}$ , is known. Multiply  $n$  and  $\bar{b}$  to form the product  $n\bar{b}$ . For the algorithm to work, this product must also be a number in the standard table. Then its reciprocal is

$$\overline{(n\bar{b})} = \bar{n}\bar{\bar{b}} = \bar{n}b.$$

Finally, multiplying by  $\bar{b}$  removes the unwanted factor,  $\bar{n}\bar{b}\bar{b} = \bar{n}$ , leaving the desired reciprocal. The crucial point in the algorithm is that  $n$  must have a regular tail  $b$  such that  $n\bar{b}$  is in the table. The example used in UET 6/2 295 is  $n = 2,5$ , one of the most common numbers used for these algorithms, split into a head  $a = 2$  and tail  $b = 5$ . The notation used in Table 2 is  $D_1$  is the input data, and  $R_k$  denotes the result of the  $k$ th step of the algorithm.

Using the data from UET 6/2 295, we have  $D_1 = 2,5$ , and the calculations are as follows.

|        | Computation                        | Symbolic Instruction     |
|--------|------------------------------------|--------------------------|
| Step 1 | Split off tail: 5                  | $R_1 = \text{tail}(D_1)$ |
| Step 2 | Reciprocal of 5: 12                | $R_2 = \overline{R_1}$   |
| Step 3 | Multiply 12 and 2,5: result 25     | $R_3 = D_1 \times R_2$   |
| Step 4 | Reciprocal of 25: 2,24             | $R_4 = \overline{R_3}$   |
| Step 5 | Multiply 2,24 and 12: result 28,48 | $R_5 = R_4 \times R_2$   |

TABLE 2. Simple Reciprocal algorithm

**Exercise 3.1.** Apply the algorithm to find the reciprocal of 1,12, using 12 as the tail. Check that this is the same as the reciprocal in the standard table.

**Exercise 3.2.** Apply the algorithm to find the reciprocal of 1,30, using 30 as the tail. Check that this is the same as the reciprocal in the standard table.

**Exercise 3.3.** Apply the algorithm to  $n = 4, 10$ , with tail 10.

**Exercise 3.4.** Attempt to apply the algorithm to  $n = 1, 10$  with tail 10. Explain what goes wrong and why.

**Exercise 3.5.** The algorithm makes two demands on the numbers used, that the tail be in the standard table, and that the intermediate number computed also be in the table. A consequence is that the algorithm can be fairly delicate. The first regular number not in the table is  $n = 1, 36$ . Use the algorithm to try to find the reciprocal of 1,36 using 36 as the tail. Explain what happens. Now repeat the process using 6 as the tail and explain the result.

#### 4. THE TECHNIQUE

The classic method of determining the reciprocal in Old Babylonian mathematics was first analyzed by A. Sachs [3], who termed the procedure, ‘The Technique’. The technique differs from the algorithm described above by splitting Step 3 into two steps, each of which is simpler computationally. Recall that Step 3 is the stage where the reciprocal of the tail is multiplied by the original number  $n$ . For example, in UET 6/2 295, the original number  $n$  is 2,5, and the tail is 5. The reciprocal of the tail is 12 and Step 3 calls for multiplying 2,5 by 12. If we view  $n$  as split into a head  $a$  and tail  $b$ , so that  $n = a + b$ , then multiplying by  $\bar{b}$ , the reciprocal of  $b$ , gives  $n\bar{b} = (a + b)\bar{b}$ . The observation underlying The Technique is that, by distributivity, we can write  $(a + b)\bar{b} = a\bar{b} + b\bar{b}$ . Since  $b$  and  $\bar{b}$  are a reciprocal pair, we always have  $b\bar{b} = 1$ . The Technique replaces the calculation of  $n\bar{b}$  by first computing  $a\bar{b}$  and then adding 1. The advantage of this method is that since  $a$  is only the head of  $n$ ,  $a\bar{b}$  is easier to compute than  $n\bar{b}$ , and that regardless of  $n$ , the second term will always be 1. The disadvantage is that it is easy to lose the sexagesimal place and add the 1 in the wrong position. This was a relatively common error in Old Babylonian school texts.

Taking  $D_1 = n = 2, 5$ , as above, The Technique gives the procedure shown in Table 3.

**Exercise 4.1.** Repeat Exercises 3.1 – 3.5 using ‘The Technique’.

Here is an example of an Old Babylonian description of using The Technique, taken from VAT 6505, probably from Sippar. The tablet is badly broken, but

|        | Computation                        | Symbolic Instruction                |
|--------|------------------------------------|-------------------------------------|
| Step 1 | Split off tail: 5                  | $R_1 = \text{tail}(D_1)$            |
| Step 2 | Reciprocal of 5: 12                | $R_2 = \overline{R_1}$              |
| Step 3 | Multiply 12 and 2: result 24       | $R_3 = \text{head}(D_1) \times R_2$ |
| Step 4 | Add 1: result 25                   | $R_4 = R_3 + 1$                     |
| Step 5 | Reciprocal of 25: 2,24             | $R_5 = \overline{R_4}$              |
| Step 6 | Multiply 2,24 and 12: result 28,48 | $R_6 = R_5 \times R_2$              |

TABLE 3. ‘The Technique’, a reciprocal algorithm

contains a list of problems on reciprocals and so the contents can be reconstructed from parallels in the different problems and knowledge of what numbers must be there. In the translation below, no indications of breaks and restorations are given (see [2, I:270] and [3]).

2,13,20. What is its reciprocal?  
 You, in your procedure,  
 find the reciprocal of 3,20. You see 18.  
 Multiply 18 by 2,10. You see 39.  
 Add 1. You see 40.  
 Find the reciprocal of 40. You see 1,30.  
 Multiply 1,30 by 18.  
 You see 27. 27 is its reciprocal.  
 Such is the procedure.

According to the colophon of the tablet, this is one of 12 problems that were originally there. However, the tablets is so broken that only portions of 5 are left. Although 27 is a reciprocal in the standard table, and thus so is 2,13,20, in this case the problems form a sequence produced by the favored process of doubling and halving. In this case, they are entries from the favorite sequence beginning with the pair 2, 5  $\sim$  28,48. The next entries are:

|          |         |
|----------|---------|
| 4,10     | 14,24   |
| 8,20     | 7,12    |
| 16,40    | 3,36    |
| 33,20    | 1,48    |
| 1,6,40   | 54      |
| 2,13,20  | 27      |
| 4,26,40  | 13,30   |
| 8,53,20  | 6,45    |
| 17,46,40 | 3,22,30 |

This sequence of pairs was often pursued quite a distance, not always accurately. For example, YBC 10802 has the pair 2, 22, 13, 20  $\sim$  25, 12, 42 (which should be 25,18,45) and MLC 651 has the pair 1, 20, 54, 31, 6, 40  $\sim$  44, 29, 37, 50, 15, 20, where it should have 44, 29, 40, 39, 50, 37, 30 (see [3]).

## REFERENCES

1. Friberg, J. (2000). "Mathematics at Ur in the Old Babylonian period," *Revue d'Assyriologie* 94, 97–188.
2. Neugebauer, O. (1935-37). *Mathematische Keilschrifttexte I-III (MKT)*. Berlin, Springer.
3. Sachs, A. (1947). "Babylonian mathematical texts: I. Reciprocals of regular sexagesimal numbers," *JCS* 1, 219–240.